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Possible Forms of Seismic Anisotropy of the Uppermost Mantle under Oceans

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Abstract. The apparent body wave velocities v_P of the oceanic mantle immediately below the Mohorovicic discontinuity (M), obtained from oceanic refraction shooting, can depend on azimuth ϕ because of real mantle anisotropy or horizontal variations in an isotropic crust and mantle. A small anisotropy of the uppermost mantle leads to

$$v_P(\phi) = \sum_{n=0}^{4} A_n \cos^n \phi \sin^{4-n} \phi = \sum_{n=-2}^{2} B_n e^{in\phi}$$

The A_n determine 5 of the 21 elastic parameters of the uppermost mantle. Curvature in the M discontinuity has the same effect on $v_P(\phi)$, but the azimuth dependence found by Hess in Raitt's and Shor's velocities, if real, is too large to be due to curvature. A sloping M discontinuity adds to $v_P(\phi)$ a term $a \cos \phi + b \sin \phi$. In a refraction survey of anisotropy, only two lines need be reversed to find a and b. If the ϕ dependence of v_P is due to a small mantle anisotropy and a small slope of the M discontinuity, $v_P(\phi)$ is determined for all ϕ by shooting five different lines through a point and reversing two of them or simply by shooting seven different lines. The theory is compared with Raitt's and Shor's data as reported by Hess.

1. *Introduction.* *Hess* [1964] has recently called attention to the possibility that anisotropy in the oceanic upper mantle is detectable in the seismic refraction work of *Raitt* [1963] near the Mendocino fracture zone and of *Shor* [1964] near Maui. The observation is that the speed of the P headwave just below the Mohorovicic discontinuity (M) appears to depend on the azimuth of the line joining shot point and receiver.

It is my purpose in the present note to indicate how an arbitrary small anisotropy in the upper mantle will affect the azimuth dependence of the measured speeds of P and S headwaves just below the M discontinuity. I hope this information may be useful in the design of experiments to study possible horizontal anisotropy in more detail.

2. *Formulation of the problem.* In principle, refraction shooting can give the azimuth dependence of the P and S wave velocities just below the M discontinuity. In practice, usually only the P wave velocity is measurable (Raitt, private communication, 1964), but it is of some interest to see what could be learned from the S wave velocities if they were available.

What is obtained by the analysis of travel-time curves in refraction shooting is the magnitude of the group velocity of the P or S body wave in the upper mantle just below the M discontinuity when the direction of that group velocity is along the line joining shot point and receiver (both shot point and receiver are assumed to be very near the surface of the ocean). Thus refraction shooting on several azimuths shows how the magnitude of the P or S wave group velocity depends on the direction of that group velocity, when the direction lies in the horizontal plane (more precisely, the plane of the M discontinuity).

The mathematical problem is therefore to determine how P and S wave group velocities vary in magnitude as their directions vary in a prescribed plane. In a general anisotropic medium this problem involves considerable computation. When, as in the present case, the medium is nearly isotropic, the problem is much simplified.

The first simplification is that, to first order in the anisotropy, the azimuth dependences of group and phase velocities are the same. To see this, consider a wave whose dispersion relation is

$$\omega = kc_0[1 + f(\boldsymbol{\nu})]$$

where ω is angular frequency, c_0 is a constant, $\mathbf{k}$ is the wave propagation vector, k is the magnitude of $\mathbf{k}$, and $\boldsymbol{\nu} = \mathbf{k}/k$. It will be assumed that $f \ll 1$. The Cartesian components G_i of the group velocity are (under the Einstein summation convention)

$$C_i(\boldsymbol{\nu}) = \frac{\partial \omega}{\partial k_i} = c_0\nu_i(1 + f) + c_0 k \frac{\partial f}{\partial \nu_j}\frac{\partial \nu_j}{\partial k_i} \quad (1)$$

Then, correct to first order in f,

$$G^2(\boldsymbol{\nu}) = G_i G_i = c_0^2\left[1 + 2f + 2k\frac{\partial f}{\partial \nu_j}\frac{\partial \nu_j}{\partial k_i}\nu_i\right]$$

But $\nu_i(\partial \nu_j/\partial k_i) = 0$, so, to first order in f,

$$G^2(\boldsymbol{\nu}) = c_0^2[1 + 2f] = c^2(\boldsymbol{\nu}) \quad (2)$$

where $c(\boldsymbol{\nu})$ is the phase velocity of waves whose propagation vector has the direction $\boldsymbol{\nu}$.

If $\boldsymbol{\nu}'$ is a unit vector in the direction of the group velocity of waves whose propagation vector has direction $\boldsymbol{\nu}$, then $\boldsymbol{\nu}' - \boldsymbol{\nu}$ is of first order in the small quantity f (see equation 1). Therefore, correct to first order in f, equation 2 implies that $G(\boldsymbol{\nu}') = G(\boldsymbol{\nu})$. In refraction shooting the phase velocity of the wave whose propagation vector points from the shot point to the receiver is measured, correct to first order in f.

The above remarks show that, correct to first order in the anisotropy, in refraction shooting phase velocity is measured as a function of direction of the propagation vector. It is my purpose in this paper to deduce the form of this function in the most general slightly anisotropic, perfectly elastic medium.

Some techniques for recognizing the arrival of the signal propagated along a particular type of ray path at various receivers involve matching the arriving wave forms. The simplicity of these techniques depends on the absence of dispersion in the body waves propagated by the various layers. A perfectly elastic anisotropic medium is nondispersive in the sense that all P waves (or S waves of type 1 or type 2) propagating in a given direction have the same phase velocity. However, the phase velocity's dependence on direction makes the medium dispersive for all but plane wave fronts. Since explosions do not generate plane wave fronts, it can be anticipated that if the mantle were very anisotropic the signal-matching technique might require prefiltering of the signals to remove dispersive distortion. The calculation leading to (2) shows that if the anisotropy is small this prefiltering is unnecessary, in the sense that omitting it leads to errors in the picking of arrival times which are of second order or smaller in the anisotropy.

3. *Perturbation-theoretic statement of the problem.* Let x_1, x_2, x_3 be Cartesian coordinates in the upper mantle just below the M discontinuity, introduce Cartesian coordinates, with the x_1 axis pointing north, the x_2 axis pointing east, and the x_3 axis pointing downward. Approximate the given region of the upper mantle as a homogeneous, anisotropic, perfectly elastic medium of density ρ. Let $\mathbf{s}$ be the displacement vector of a body wave in the medium; the angular frequency of the wave is ω, and its propagation vector is $\mathbf{k}$, while $\boldsymbol{\nu} = \mathbf{k}/k$ is a unit vector in the direction of $\mathbf{k}$. The strain tensor is

$$\sigma_{ij} = \frac{1}{2}\left(\frac{\partial s_i}{\partial x_j} + \frac{\partial s_j}{\partial x_i}\right)$$

and the stress tensor is T_{ij}. The stress-strain relation is

$$T_{ij} = \rho\Gamma_{ijkl}\sigma_{kl} \quad (3)$$

where $\rho\Gamma_{ijkl}$ is the elastic tensor. Since $\Gamma_{ijkl} = \Gamma_{jikl} = \Gamma_{ijlk} = \Gamma_{klij}$, this tensor has at most 21 independent components.

The equations of motion are $\rho\partial^2 s_i/\partial t^2 = \partial T_{ij}/\partial x_j$. For the body wave in question, these equations simplify to $\omega^2 s_i = k_j k_k \Gamma_{ijkl}\, s_l$. Let $c = \omega/k$ be the phase velocity of the body wave. Then

$$(\Gamma_{ijkl}\nu_j\nu_k)s_l = c^2 s_i \quad (4)$$

Equation 4 constitutes an eigenvalue problem. Let

$$B_{il} = \Gamma_{ijkl}\nu_j\nu_k \quad (5)$$

$$B = c^2 \quad (6)$$

Then equation 4 is

$$B_{il}s_l = Bs_i \quad (7)$$

The three eigenvalues B of the symmetric 3×3 matrix B_{il} are the squared phase velocities of the three different body waves whose wave vector has the direction $\boldsymbol{\nu}$. The three corresponding eigenvectors, $s_i^{(1)}$, $s_i^{(2)}$, and $s_i^{(3)}$ give the polarizations of these three body waves.

In the special case of a nearly isotropic medium,

$$\Gamma_{ijkl} = \Gamma_{ijkl}^{(0)} + \gamma_{ijkl} \quad (8)$$

where

$$\Gamma_{ijkl}^{(0)} = (c_P^2 - 2c_S^2)\delta_{ij}\delta_{kl} + c_S^2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad (9)$$

and $\gamma_{ijkl} \ll c_S^2$. That is, Γ_{ijkl} is the sum of an isotropic tensor $\Gamma_{ijkl}^{(0)}$ and a much smaller tensor expressing the anisotropy of the medium. The isotropic tensor $\Gamma_{ijkl}^{(0)}$ describes an isotropic medium with P wave phase velocity c_P and S wave phase velocity c_S. For the isotropic medium,

$$B_{il}^{(0)} = (c_P^2 - c_S^2)\delta_{il} + c_S^2\nu_i\nu_l \quad (10)$$

and in the slightly anisotropic medium

$$B_{il} = B_{il}^{(0)} + b_{il}$$

where

$$b_{il} = \gamma_{ijkl}\nu_j\nu_k \quad (11)$$

If $b_{il} = 0$, the solutions of (7) are well known. There is a nondegenerate eigenvalue $B = c_P^2$, with unit eigenvector $s_i^P = \nu_i$, and there is a degenerate eigenvalue $B = c_S^2$ whose two-dimensional space of eigenvectors consists of all vectors s_i that are perpendicular to $\mathbf{v}$.

When $b_{il} \neq 0$, the solution of (7) can be written in the form

$$B = B^{(0)} + B^{(1)} + B^{(2)} + \cdots \quad (12)$$

$$s_i = s_i^{(0)} + s_i^{(1)} + s_i^{(2)} + \cdots \quad (13)$$

where $B^{(0)}$ and $s_i^{(0)}$ are a solution of (7) with $b_{il} = 0$ and $B^{(n)}$ and $s_i^{(n)}$ are of nth order in b_{il}. If $B^{(0)}$ is nondegenerate, and $s_i^{(0)}$ is chosen to be a unit vector,

$$B^{(1)} = b_{il}s_i^{(0)}s_l^{(0)} \quad (14)$$

If $B^{(0)}$ is degenerate with a two-dimensional eigenspace and $s_i^{(SH)}$ and $s_i^{(SV)}$ are two mutually orthogonal unit vectors satisfying $B_{il}^{(0)}s_l^{(0)} = B^{(0)}s_i^{(0)}$, then in the expansion (13), $s_i^{(0)} = \alpha s_i^{(SH)} + \beta s_i^{(SV)}$ and

$$\begin{bmatrix} s_i^{(SH)}b_{il}s_l^{(SH)} & s_i^{(SH)}b_{il}s_l^{(SV)} \\ s_i^{(SV)}b_{il}s_l^{(SH)} & s_i^{(SV)}b_{il}s_i^{(SV)} \end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = B^{(1)}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} \quad (15)$$

4. *Solution for P waves.* From the foregoing remarks it follows that the P wave in the isotropic medium has in the anisotropic medium the velocity

$$v_P^2 = c_P^2 + B^{(1)} \text{ plus terms of second order in } \gamma_{ijkl} \quad (16)$$

where, since $s_i^P = \nu_i$,

$$B^{(1)} = \gamma_{ijkl}\nu_i\nu_j\nu_k\nu_l \quad (17)$$

Let ϕ be azimuth, measured east of north in the coordinate system introduced at the beginning of section 3. Then for the P headwaves excited in refraction shooting

$$(\nu_1, \nu_2, \nu_3) = (\cos\phi, \sin\phi, 0) \quad (18)$$

and, correct to first order in γ_{ijkl},

$$Q(\phi) \equiv v_P^2(\phi) - c_P^2 = \gamma_{1111}\cos^4\phi + 4\gamma_{1112}\cos^3\phi\sin\phi + (2\gamma_{1122} + 4\gamma_{1212})\cos^2\phi\sin^2\phi + 4\gamma_{1222}\cos\phi\sin^3\phi + \gamma_{2222}\sin^4\phi \quad (19)$$

That is, the deviation $Q(\phi) = v_P^2(\phi) - c_P^2$ is a homogeneous trigonometric polynomial of degree 4 in ϕ. The higher-order terms of (12) and (13) will contribute to (19) trigonometric polynomials of degree larger than 4. These correction terms are of the order of magnitude $\gamma_{ijkl}\,\gamma_{ijkl}/c_S^2$ and are small when the anisotropy is small (i.e., $\gamma_{ijkl} \ll c_S^2$).

It is easy to verify that a homogeneous trigonometric polynomial of degree n,

$$q(\phi) = \sum_{m=0}^{n} A_m \cos^m\phi \sin^{n-m}\phi \quad (20)$$

has the following properties:

(1) The coefficients $A_0, \cdots, A_n$ are uniquely determined by the values of $q(\phi)$ at any $n + 1$ angles ϕ satisfying $0 \leq \phi_0 < \phi_1 < \cdots < \phi_n < \pi$.

(2) The Fourier series for $q(\phi)$ is

$$q(\phi) = \sum_{m=0}^{[n/2]} [B_m \cos(n - 2m)\phi + C_m \sin(n - 2m)\phi]$$

where $[n/2]$ is the largest integral m which permits $n - 2m$ to be non-negative.

Property (2) is established by induction and property (1) follows from

$$q(\phi) = \cos^n \phi \sum_{m=0}^{n} A_m (\tan \phi)^m$$

That is, $\sec^n \phi q(\phi)$ is a polynomial of degree n in $\tan \phi$, and is determined by its values at any $n + 1$ different values of $\tan \phi$.

For (19), the Fourier series is

$$Q(\phi) = A + C \cos 2\phi + D \sin 2\phi + E \cos 4\phi + F \sin 4\phi \qquad (21)$$

where

$$8A = 3\gamma_{1111} + 2\gamma_{1122} + 4\gamma_{1212} + 3\gamma_{2222}$$
$$2C = \gamma_{1111} - \gamma_{2222}$$
$$D = \gamma_{1112} + \gamma_{1222} \qquad (22)$$
$$8E = \gamma_{1111} - 2\gamma_{1122} - 4\gamma_{1212} + \gamma_{2222}$$
$$2F = \gamma_{1112} - \gamma_{1222}$$

The conclusion is that in refraction shooting over a slightly anisotropic mantle the Fourier expansion of the square of the M discontinuity P wave velocity v_P as a function of azimuth ϕ contains only five terms: 1, $\cos 2\phi$, $\sin 2\phi$, $\cos 4\phi$ and $\sin 4\phi$. A different dependence on ϕ, if not large, has some cause other than anisotropy. Furthermore, the anisotropy in v_P is not determined unless at least five noncollinear azimuths are shot, and any five are sufficient to determine this anisotropy completely. Finally, the anisotropy in v_P depends only on the elastic constants γ_{1111}, γ_{2222}, γ_{1222}, γ_{1112}, and $\gamma_{1122} + 2\gamma_{1212}$; the anisotropy in v_P determines these five constants uniquely.

5. *Solution for S waves.* For the S headwaves in refraction shooting, if ν_i are given by (18), then

$$s_i^{(SH)} = -\epsilon_{im3}\nu_m \qquad (23)$$

ϵ_{ijk} being the three-dimensional alternating tensor, and

$$s_i^{(SV)} = \delta_{i3} \qquad (24)$$

Moreover,

$$\epsilon_{im3}\epsilon_{ln3}\nu_m\nu_n = \delta_{il} - \delta_{i3}\delta_{l3} - \nu_i\nu_l \qquad (25)$$

If v_S is the velocity of the S headwaves, $v_S^2 - c_S^2 = B^{(1)}$ has two possible values, the two eigenvalues of the 2×2 matrix on the left of (15). Let $B_1^{(1)}(\phi)$ and $B_2^{(1)}(\phi)$ be those two eigenvalues. Then $T(\phi) = B_1^{(1)} + B_2^{(1)}$ is the trace of that matrix, and $\Delta(\phi) = B_1^{(1)}B_2^{(1)}$ is the determinant of the matrix. Thus

$$T(\phi) = \gamma_{ijki}\nu_j\nu_k - \gamma_{ijkl}\nu_i\nu_j\nu_k\nu_l \qquad (26)$$

and

$$\Delta(\phi) = (\gamma_{ijk3}\gamma_{lrs3} - \gamma_{ijkl}\gamma_{3rs3} + \gamma_{mjkm}\gamma_{3rs3}\delta_{il} - \gamma_{mjk3}\gamma_{mrs3}\delta_{il})\nu_i\nu_j\nu_k\nu_l\nu_r\nu_s \qquad (27)$$

Let $v_{S1}(\phi)$ and $v_{S2}(\phi)$ be the two observed S wave velocities for body waves with horizontal propagation vectors. Then from (16), (17), (18), (19), and (26), $v_{S1}^2(\phi) + v_{S2}^2(\phi) + v_P^2(\phi) - c_P^2 - 2c_S^2$ is

$$T(\phi) + Q(\phi) = \gamma_{ijki}\nu_j\nu_k = \gamma_{i11i} \cos^2 \phi + 2\gamma_{i12i} \cos \phi \cdot \sin \phi + \gamma_{i22i} \sin^2 \phi \qquad (28)$$

and $[v_{S1}^2(\phi) - c_S^2][v_{S2}^2(\phi) - c_S^2]$ is

$$\Delta(\phi) = \sum_{m=0}^{6} B_m \cos^m \phi \sin^{6-m} \phi \qquad (29)$$

where the B_m are sums of products of pairs of the γ_{ijkl}.

It follows that $T(\phi) + Q(\phi)$ is a homogeneous trigonometric polynomial of second degree in the azimuth ϕ and that $\Delta(\phi)$ is a homogeneous trigonometric polynomial of degree 6 in ϕ. Therefore $T(\phi) + Q(\phi)$ is completely determined for all ϕ by shooting on any three noncollinear azimuths, whereas determing $\Delta(\phi)$ requires seven noncollinear azimuths. If $T(\phi)$ and $\Delta(\phi)$ are known, the two S wave velocities v_{S1} and v_{S2} in the horizontal direction ϕ can be determined, since $v_{S1}^2 - c_S^2$ and $v_{S2}^2 - c_S^2$ are the two roots λ of the equation

$$\lambda^2 - T\lambda + \Delta = 0 \qquad (30)$$

If $T(\phi) + Q(\phi)$ is known, three more elastic constants besides those appearing in (22) are determined, namely $\gamma_{1111} + \gamma_{2112} + \gamma_{3113}$, $\gamma_{1121} + \gamma_{2122} + \gamma_{3123}$, and $\gamma_{1221} + \gamma_{2222} + \gamma_{3223}$. Therefore, if v_P^2, v_{S1}^2, and v_{S2}^2 can be measured, the following eight elastic constants are determined: γ_{1111}, γ_{1112}, γ_{1222}, γ_{2222}, $\gamma_{1122} + 2\gamma_{1212}$, γ_{1323}, $\gamma_{1313} + \gamma_{1212}$, and $\gamma_{1212} + \gamma_{2323}$. In addition, seven more quadratic expressions in γ_{ijkl} are determined, namely the seven Fourier coefficients of the sixth-order trigonometric polyno-

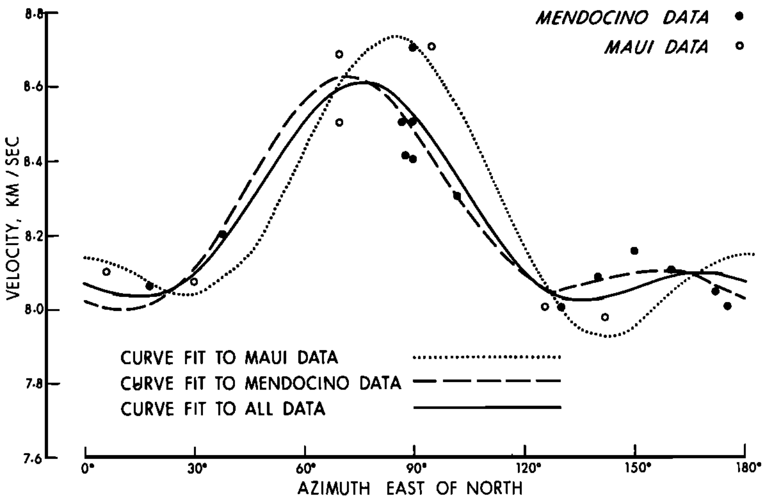

Fig. 1. Uppermost mantle velocity versus azimuth for refraction lines near the Mendocino fracture zone and Maui. Data from Raitt and Shor.

mial (29). Therefore, measuring v_P^2, v_{S1}^2, and v_{S2}^2 on seven noncollinear azimuths will give all the information about mantle anisotropy that is obtainable from measurements of the travel times of headwaves just below the M discontinuity. This information consists of 15 relations among the 21 elastic coefficients. It was not expected that from travel times in surface refraction shooting all the anisotropic elastic coefficients could be completely determined; it is perhaps surprising that travel times in a single plane determine as many as 15 relations among the 21 elastic parameters.

6. *Comparison with observation.* In Figure 1 are plotted Raitt's data for $v_P(\phi)$ from the Mendocino fracture zone and Shor's data from near Maui. The shot locations are individually mapped by *Hess* [1964]. Curves of form 21 which fit Raitt's data, Shor's data, and both sets of data regarded as a single population are also drawn in Figure 1. The curves are fitted to the data by least squares. The equations of the curves are as follows, in km² sec⁻². Mendocino: $v_P^2(\phi) = 67.722 + 2.336 \sin 2\phi - 3.806 \cos 2\phi - 2.163 \sin 4\phi + 0.492 \cos 4\phi$; Maui: $v_P^2(\phi) = 67.90 + 1.698 \sin 2\phi - 4.786 \cos 2\phi - 0.876 \sin 4\phi + 3.130 \cos 4\phi$; Mendocino and Maui together: $v_P^2(\phi) = 67.663 + 2.100 \sin 2\phi - 3.796 \cos 2\phi - 1.677 \sin 4\phi + 1.205 \cos 4\phi$.

The elastic parameters corresponding to these three curves are given in Table 1. In the computations it has been assumed that $c_P^2 = 67.75$ km² sec⁻².

The Maui elastic parameters should not be given much credence, since only seven data points were available to determine five parameters. From Figure 1 it is clear that the data are consistent with the hypothesis of mantle anisotropy; the fit is as good as the internal consistency of the data. From Table 1 it is clear that the data available at the present time do not warrant the conclusion that the anisotropy is the same at Maui and Mendocino. In fact, a large part of the scatter in Figure 1 may be the result of geographical variations in the anisotropy.

7. *Effects of slight deviations from horizontal stratification.* In the foregoing discussion it has been presupposed that the variations, if any, in

TABLE 1. Elastic Parameters of Uppermost Mantle Computed from Anisotropies in Refraction Shooting

Elastic Parameter	Mendocino Value, km² sec⁻²	Maui Value, km² sec⁻²	Combined Value, km² sec⁻²
γ_{1111}	−3 344	−1 506	−2.681
γ_{2222}	4.268	8.066	4.911
$\gamma_{1122} + 2\gamma_{1212}$	−1.506	−9.540	−3.705
γ_{1112}	−0.995	−0.027	−0 627
γ_{1222}	3.331	1.725	2.727

density and elastic parameters above the M discontinuity depend only on x_3 and that the M discontinuity is parallel to the $x_1 - x_2$ plane. We will now consider whether the apparent anisotropy noticed by Hess in Raitt's and Shor's travel times can be the result of a small tilt or warping of the M discontinuity or the overlying structure rather than of a real anisotropy in the material of the uppermost mantle.

It is possible, and perhaps likely, that there are horizontal variations in the crust with length scales considerably smaller than the length L of the longest line shot in Raitt's and Shor's surveys (L was about 100 km); but it is difficult to see how such small-scale variations could produce a systematic apparent anisotropy. A likelier source of systematic apparent anisotropy would be horizontal variations with length scales of the order of or greater than L. For the purposes of the present discussion suppose that within any horizontal circle of radius L the M discontinuity can be well approximated by an equation of the form

$$x_3 = D + \eta_1 x_1 + \eta_2 x_2 + \tfrac{1}{2}\kappa_{11} x_1^2 + \kappa_{12} x_1 x_2 + \tfrac{1}{2}\kappa_{22} x_2^2 \qquad (31)$$

where D, η_i, κ_{ij} are constants, and η_1, η_2, $\kappa_{11} L$, $\kappa_{12} L$, and $\kappa_{22} L$ are all much less than 1. Suppose also that the crust and mantle are isotropic and that the real P wave velocity c_P depends slightly on x_1 and x_2, so that in any horizontal circle of radius L, to a good approximation,

$$c_P^{-1}(x_1, x_2, x_3) = c_0^{-1}(x_3) + \alpha_1(x_3)x_1 + \alpha_2(x_3)x_2 + \tfrac{1}{2}\beta_{11}(x_3)x_1^2 + \beta_{12}(x_3)x_1x_2 + \tfrac{1}{2}\beta_{22}(x_3)x_2^2 \qquad (32)$$

Suppose that points P, Q_1, Q_2, $\cdots$, Q_n on the surface of the ocean are collinear and that P does not lie between any of the Q's. Then the directed straight-line segment from P to the most distant Q is a 'shot line' with 'anchor point' P. The azimuth ϕ of the shot line will be measured positive east of north. The apparent velocity $v_{MA}(\phi)$ of a headwave traveling on azimuth ϕ in the mantle just below the M discontinuity is measured by measuring the travel times (from P to the various Q's) of the signal whose ray descends from P directly to the M discontinuity, travels as the required headwave just below the M discontinuity, and ascends directly from the M discontinuity to one of the Q's. As the point Q is displaced along the shot line with anchor point P and azimuth ϕ, the travel time T and the horizontal distance S between P and Q both vary. The apparent velocity $v_{MA}(\phi)$ is measured as

$$v_{MA}(\phi) = dS/dT \qquad (33)$$

If the deviations from horizontal stratification described by (31) and (32) are small, first-order perturbation theory is adequate for calculating $v_{MA}(\phi)$. The details of the calculations are given in the appendix.

Consider first the effect of a sloping, warped M discontinuity, as described by (31). Anisotropy and horizontal variations in velocities, as well as tilt and curvature of crustal bedding planes and the ocean bottom, are, for the moment, neglected. It is shown in the appendix that in this case, to first order in η_1, η_2, $\kappa_{11} L$, $\kappa_{12} L$, and $\kappa_{22} L$,

$$v_{MA}(\phi)v_M^{-1} = 1 - \cot\tau(\eta_1 \cos\phi + \eta_2 \sin\phi) - (R + \rho)\cot\tau(\kappa_{11}\cos^2\phi + 2\kappa_{12}\cos\phi\sin\phi + \kappa_{22}\sin^2\phi) \qquad (34)$$

where v_M is the P wave velocity in the mantle just below the M discontinuity, τ is the critical angle of refraction at the M discontinuity, and $R + \rho$ is the horizontal distance from the anchor point of the shot line to the point on the M discontinuity where the ray is refracted back into the crust (point N' in Figure 2).

The anisotropy produced by the slope of the M discontinuity, η_1 and η_2, has a different azimuth dependence from that produced by a real anisotropy and cannot explain the regularity Hess finds in Raitt's and Shor's data. The anisotropy produced by a curvature of the M dis-

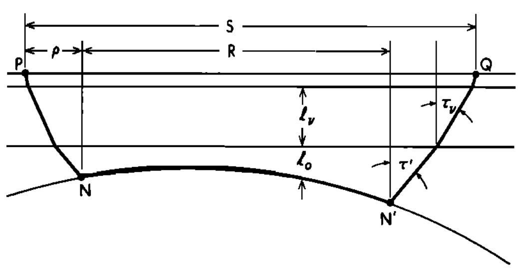

Fig. 2. The path used to calculate travel time from P to Q when the only deviation from horizontal stratification is a sloping, curved M discontinuity. A layered crust is shown.

continuity, κ_{11}, κ_{12}, and κ_{22}, has a Fourier series consisting of a constant plus the term

$$(\kappa/2)(R + \rho) \cot \tau \cos (2\phi + \psi_0)$$

where $\kappa = [(\kappa_{11} - \kappa_{22})^2 + 4\kappa_{12}^2]^{1/2}$ and ψ_0 is a constant. This term has the ϕ dependence observed in Raitt's and Shor's data. Yet there are two objections to the hypothesis that M discontinuity curvature produces the observed anisotropy. First, curvature makes $v_{MA}(\phi)$ dependent on the length of the shot line to the same extent that it is dependent on azimuth, at least for lines long in comparison with the depth of the M discontinuity. Second, even if the shot lines are so short that this effect is below noise level in the data, the observed anisotropy in Raitt's and Shor's data requires $\kappa(R + \rho) \cot \tau = 8 \times 10^{-2}$ km^{-1}. With $\cot \tau = 1.4$ and $R + \rho = 100$ km (an upper limit on the length of Raitt's shot lines), κ is 5.7×10^{-4} km^{-1}. The area surveyed in Raitt's measurements is roughly a rectangle 1600 km from east to west and 1000 km from north to south. (Shor's surveyed area was smaller.) If κ were as large as 5.7×10^{-4} km^{-1}, the depth to the M discontinuity would vary by 70 km in the surveyed area. This variation is not observed. An alternative explanation for such a highly curved M discontinuity might be that the curvatures were large but changed sign from place to place in the surveyed area. *Mason*'s [1958] magnetic maps make a corrugated M discontinuity not completely inconceivable. Yet, except for three anomalously low velocities in a disturbed area near the coast just north of the Mendocino escarpment, which Raitt and Hess agree probably belong to a different geographical province from the other measurements, Raitt and Shor found every east-west mantle velocity larger than any north-south mantle velocity. The sign of the anisotropy is determined by the difference between the east-west and north-south curvatures, and it seems improbable that all of Raitt's and Shor's measurements were made where that sign happened to be negative.

Now consider the effect of deviation from horizontal stratification in the crust and mantle as described by (32), the M discontinuity being a plane parallel to the x_1-x_2 plane. Equation 32 applies to horizontal variations in velocities, as well as to tilts and curvatures of the ocean bottom and any crustal bedding planes other than the M discontinuity. Such tilts and curvatures appear as Dirac delta functions of small amplitude and variable depth in the functions $\alpha_i(x_3)$ and $\beta_{ij}(x_3)$. It is shown in the appendix that, if the crust and mantle are isotropic, the M discontinuity is the horizontal plane $x_3 = D$, and the horizontal variations in crust and mantle are given by (32), then to first order in the small quantities

$$\begin{aligned} v_M L\langle|\alpha_i|\rangle &= v_M L D^{-1} \int_0^D |\alpha_i(x_3)| \, dx_3 \\ v_M L^2\langle|\beta_{ij}|\rangle &= v_M L^2 D^{-1} \int_0^D |\beta_{ij}(x_3)| \, dx_3 \end{aligned} \tag{35}$$

The apparent mantle velocity obtained from travel-time curves is given by

$$\begin{aligned} v_{MA}(\phi)v_M^{-1} = 1 &- A_1 \cos \phi \\ &- A_2 \sin \phi - B_{11} \cos^2 \phi \\ &- 2B_{12} \cos \phi \sin \phi - B_{22} \sin^2 \phi \end{aligned} \tag{36}$$

The coefficients in (36) are defined as follows: if $R + \rho$ has the same meaning as in (34), l is arc length along the ray, and $s(l)$ is the horizontal distance from the point N' in Figure 2 to the point of integration, then

$$A_i = v_M(R + \rho)\alpha_i(D) + v_M \int_{N'}^{Q} \alpha_i(x_3) \, dl \tag{37}$$

and

$$B_{ij} = \tfrac{1}{2}v_M(R + \rho)^2\beta_{ij}(D) + v_M \int_{N'}^{Q} \beta_{ij}(x_3)[R + \rho + s(l)] \, dl \tag{38}$$

the integrals being taken along the ray from N' to Q, and $\alpha_i(D)$ and $\beta_{ij}(D)$ referring to the uppermost mantle just below the M discontinuity. The first term on the right in (37) and (38) represents the effect of a possible horizontal variation of velocity in the mantle just below the M discontinuity. Among short lines with a given anchor point P and different azimuths, the mantle headwaves travel through different parts of the uppermost mantle with different mean velocities. The second terms in (37) and (38) give the effects of horizontal variations in the ocean and crust.

Horizontal variations of velocity in the mantle can be dismissed as the cause of the apparent anisotropy in Raitt's and Shor's data. Those data would require variations of 4% in 100 km. If this variation were uniform over Raitt's surveyed rectangle, it would amount to at least 40% across the rectangle. If the variation changed sign from place to place, it would have to be assumed that all of Raitt's and Shor's measurements happened to be made where the variation was of the same sign.

The first terms in (37) and (38), representing horizontal variations of velocity in the mantle, will be neglected. The A_i produce a ϕ dependence other than that observed, and they will also be neglected. It remains to consider the second term on the right in (38). When $A_1 = A_2 = 0$, the Fourier expansion of the right-hand side of (36) is a constant plus $\frac{1}{2}B \cos(2\phi + \psi_0)$, where ψ_0 is a constant and $B = [(B_{11} - B_{22})^2 + 4B_{12}^2]^{1/2}$. If the observed anisotropy is to be explained by (36), B must be 0.04. The amplitude of the relative variation in mean velocity in the crust over a rectangle whose smaller side is Y is at least $Y^2 B[16l(R + \rho)]^{-1}$, which in the present case amounts to 250% and is quite inadmissible.

In conclusion, the apparent anisotropy found by Hess in Raitt's and Shor's data cannot be explained by horizontal variations in an isotropic crust and mantle unless it is postulated that by chance all of Raitt's and Shor's shot lines happened to be located on independent local features, no larger than a shot line in diameter, each of which gave a larger east-west than north-south apparent mantle velocity. For example, if the M discontinuity were corrugated with north-south ridges and valleys, it would be necessary to assume that all the shot lines lay above valleys and none lay above ridges.

8. *Conclusions.* If a homogeneous perfectly elastic medium is nearly an isotropic medium with a P wave velocity c_P and an S wave velocity c_S, and if errors of second order in the small anisotropy are neglected, the travel time of a body wave along a straight path is the path length divided by the phase velocity of that wave when its propagation vector points along the path.

If $v_P(\phi)$, $v_{S1}(\phi)$, and $v_{S2}(\phi)$ are the phase velocities of the P wave and the two S waves whose propagation vector lies in the $x_1 - x_2$ plane and makes an angle ϕ with the x_1 axis, and if

$$Q(\phi) = v_P^2(\phi) - c_P^2 \tag{39}$$

$$T(\phi) = v_{S1}^2(\phi) + v_{S2}^2(\phi) - 2c_S^2 \tag{40}$$

$$\Delta(\phi) = (v_{S1}^2(\phi) - c_S^2)(v_{S2}^2(\phi) - c_S^2) \tag{41}$$

then $T(\phi) + Q(\phi)$ is a homogeneous trigonometric polynomial in ϕ of second degree, $Q(\phi)$ is a homogeneous trigonometric polynomial of fourth degree, and $\Delta(\phi)$ is a homogeneous trigonometric polynomial of sixth degree.

In principle, $Q(\phi)$, $T(\phi)$, and $\Delta(\phi)$ can be measured for the horizontal plane in the upper mantle just below the M discontinuity by refraction with shot point and receiver near the surface of the ocean. If the crust and upper mantle are horizontally uniform, the values of $Q(\phi)$ and $v_P(\phi)$ are determined for all ϕ by measurements for five ϕ representing noncollinear shot lines, since $Q(\phi)$ has the form of (19). Measurement of $v_{S1}(\phi)$ and $v_{S2}(\phi)$ for three noncollinear shot lines determines $T(\phi) + Q(\phi)$ for all ϕ, on account of (28). And measurement of $v_{S1}(\phi)$ and $v_{S2}(\phi)$ for seven noncollinear shot lines determines $\Delta(\phi)$ for all ϕ, on account of (29). Then measurement of $v_{S1}(\phi)$ and $v_{S2}(\phi)$ for seven noncollinear shot lines determines those velocities for all ϕ, since they are the two roots λ of (30).

The dependence of $Q(\phi)$ on azimuth determines 5 of the 21 elastic coefficients of the upper mantle just below the M discontinuity. The dependence of $T(\phi) + Q(\phi)$ on ϕ determines 3 more, and the dependence of $\Delta(\phi)$ on ϕ gives 7 quadratic relations among the elastic coefficients.

If the M discontinuity is plane but gently sloping, or if the crust varies horizontally in such a way that the variation is small and linear over the length of a shot line, the values of $Q(\phi)$, $T(\phi) + Q(\phi)$, and $\Delta(\phi)$ measured by refraction shooting will differ from expressions (19), (28), and (29) by terms of the form $a \cos \phi + b \sin \phi$. The additional Fourier coefficients in these terms can be determined by shooting two more lines (which may be reversals of lines already shot).

Curvature in the M discontinuity would produce an apparent anisotropy in $v_P(\phi)$ which has the same angular dependence as a real material anisotropy. However, the curvature required to produce an anisotropy of the magnitude which Hess finds in Raitt's and Shor's data is approximately 5.7×10^{-4} km^{-1} (in excess of the earth's surface curvature). Such a large curvature, if uniform over the area surveyed by Raitt, would entail a variation of 70 km in the depth to the M discontinuity within the surveyed area. The alternative, that the M discontinuity is corrugated, with a curvature large but of variable sign, is an explanation of the anisotropy in Raitt's data if all of Raitt's fourteen shot lines happened to lie on corrugations with negative curvature. On the basis of the presently available data, it appears unlikely that the anisotropy observed in Raitt's and Shor's data can be due to curvature in the M discontinuity. Essentially the same arguments also exclude the possibility that the observed anisotropy might be due to quadratic horizontal variations of velocity in the crust or mantle or in the variations of ocean depth, even though such variations would produce the observed azimuth dependence of apparent mantle velocity.

It would be very interesting to measure $v_{MA}(\phi)$ on ten or more noncollinear azimuths at a single station (probably reversing the profiles), to see if a better fit to the theoretical curve (19) could be obtained.

From the refraction data now available not much can be said about the cause of the anisotropy, if it exists. *Hess* [1964] has suggested that it is produced by alignment of olivine crystals. Alternatively, it might result from a static stress pattern now present in an isotropic material. To verify whether such a hypothesis could explain the magnitude of the anisotropies observed, even permitting the deviator of the stress tensor to be as large as the yield stress, would require theoretical or experimental information about the quadratic terms in the upper mantle's stress-strain relation.

Appendix

To calculate $v_{MA}(\phi)$ when the M discontinuity has the configuration of (31) but there are no other horizontal variations in the crust and mantle, construct a vertical plane p^* containing the shot line with anchor point P and azimuth ϕ. The ray r^* whose travel time from P to Q is stationary subject to the constraint that it lie in p^* is not the true ray joining P and Q. Its location and direction differ from those of the true ray by amounts of the order of the small quantities η_i and $\kappa_{ij}L$. But the true ray is a path of stationary time, so that the travel time on r^* differs from that on the true ray by a quantity of second order in η_i and $\kappa_{ij}L$. This second-order error will be neglected so that the travel time on the true ray can be equated to the travel time on r^*. Now if c^*, the curve of intersection of p^* and the M discontinuity is concave downward, there will be many rays in p^* joining P and Q which have stationary travel time (subject to the constraint that they lie in p^*). One of these rays will arrive at c^* at the critical angle of refraction and will travel along c^*; the others will arrive at c^* with angles of refraction slightly less than critical, will enter the mantle as straight lines, will perhaps be totally reflected several times from the M discontinuity, and finally will leave the mantle with angles of refraction slightly less than critical. Fortunately, the question of which of these infinitely many rays should be called r^* and used to compute $v_{MA}(\phi)$ need not be discussed; correct to first order in η_i and $\kappa_{ij}L$, all rays in question have the same travel time. To prove this, consider Figure 3. All the infinitely many rays in question lie between the ray $PNN'Q$ and the ray $PFF'Q$. Stratification in the crust, being irrelevant, is not shown. Construct FG perpendicular to PN, and NM perpendicular to FF'. Let τ be the critical angle of refraction at the M discontinuity, and v_C the P wave velocity just above the discontinuity, so that $\sin \tau = v_C/v_M$. The length (FN) is of first order in η_i and $\kappa_{ij}L$, since if $\eta_i = 0$ and $\kappa_{ij} = 0$, then $(FN) = 0$. Therefore, correct to first order in these small quantities, $(GN) = (FN) \sin \tau$ and $(FM) = (FN)$. Thus $(GN)v_C^{-1} = (FM)v_M^{-1}$. Also, correct to first order, $(PF) = (PG)$, $(QF') = (QG')$, and $(G'N')\, v_C^{-1} = (F'M')v_M^{-1}$. Finally, to first order the arc NN' has the same length as the straight line segment MM'. Thus the travel times on $PNN'Q$ and $PFF'Q$ agree to first order. The same argument applies to

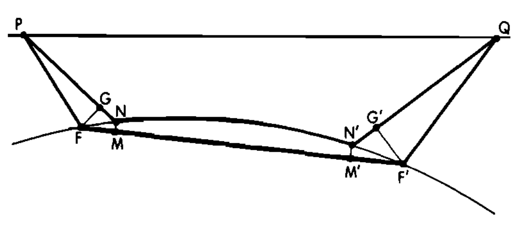

Fig. 3. Two of the infinity of paths joining P and Q for which the travel time is stationary, subject to the constraint that the paths lie in the (vertical) plane of the figure. Inhomogeneity of the crust is not shown.

the intermediate rays not shown in Figure 3. Therefore we can calculate $v_{MA}(\phi)$ from the ray $PNN'Q$, which ray will be taken to be r^*.

To summarize, $v_{MA}(\phi)$ can always be calculated correctly to first order in η_i and $\kappa_{ij}L$ by assuming that the signal has traveled in the plane p^*, obeying Snell's law as it descends from P to the M discontinuity, traveling along the intersection of p^* and the M discontinuity, and obeying Snell's law again as it rises from the M discontinuity to Q.

To calculate $v_{MA}(\phi)$, draw vertical lines through N and N' (see Figure 2). As the point Q moves to the right, the horizontal distance R between these lines changes by an amount δR, and the angle τ' between QN' and the vertical changes by an amount $\delta\tau'$, while the depth of N' below the surface PQ increases by δx_3. We regard R rather than S as the independent variable. Suppose the crust and ocean can be divided into $n + 1$ homogeneous layers (the argument is the same for a continuously varying crust, but sums are replaced by integrals), the thickness of the νth layer being l_ν, its P velocity being v_ν, and the inclination of the ray $N'Q$ to the vertical in the νth layer being τ_ν. Let l_0 be the thickness of the bottom layer. Then l_0 depends on R, but $l_1, \cdots, l_n$ and $v_0, v_1, \cdots, v_n$ do not. If R increases by δR, the change in S is

$$\delta S = \delta R + \delta\left(\sum_{\nu=0}^{n} l_\nu \tan \tau_\nu\right)$$

$$= \delta R + \delta x_3 \tan \tau' + \sum_{\nu=0}^{n} l_\nu \sec^2 \tau_\nu \, \delta\tau_\nu$$

The change in the travel time is, to first order in η_i and $\kappa_{ij}L$,

$$\delta T = v_M^{-1}\delta R + \delta\left(\sum_{\nu=0}^{n} v_\nu^{-1} l_\nu \sec \tau_\nu\right)$$

$$= v_M^{-1}\delta R + v_C^{-1} \sec \tau' \delta x_3 + \sum_{\nu=0}^{n} v_\nu^{-1} l_\nu \sec \tau_\nu \tan \tau_\nu \delta\tau_\nu$$

Since $v_\nu^{-1} \sin \tau_\nu = v_C^{-1} \sin \tau'$,

$$v_\nu^{-1} \cos \tau_\nu \delta\tau_\nu = v_C^{-1} \cos \tau' \delta\tau'$$

Thus

$$\delta S = \delta R + \delta x_3 \tan \tau' + \delta\tau \cot \tau' \Sigma$$

and

$$v_M \delta T = \delta R + \delta x_3 \sec \tau' \csc \tau + \delta\tau' \csc \tau \cos \tau' \Sigma$$

where

$$\Sigma = \sum_{\nu=0}^{n} l_\nu \sec^2 \tau_\nu \tan \tau_\nu$$

Therefore

$$\frac{dS}{dR} = 1 + \frac{dx_3}{dR} \tan \tau' + \frac{d\tau'}{dR} \cot \tau' \Sigma$$

and

$$v_M \frac{dT}{dR} = 1 + \frac{dx_3}{dR} \sec \tau' \csc \tau + \frac{d\tau'}{dR} \cos \tau' \csc \tau \Sigma$$

Since dx_3/dR and $d\tau'/dR$ are of first order in $\kappa_{ij}L$ and η_i correct to that order

$$v_M^{-1} \frac{dS}{dT} = 1 + \frac{dx_3}{dR} (\tan \tau' - \sec \tau' \csc \tau) + \frac{d\tau'}{dR} (\cot \tau' - \cos \tau' \csc \tau)\Sigma$$

On the right in the foregoing equation setting $\tau = \tau'$ produces only a second-order error, so finally, from (33),

$$\frac{v_{MA}(\phi)}{v_M} = 1 - \frac{dx_3}{dR} \cot \tau \qquad (39)$$

which is correct to first order in η_i and $\kappa_{ij}L$. It is interesting that the variation of τ' with R cancels out of the final result, leaving only the direct effect of the variation in the depth to the M discontinuity.

There remains only to set $x_1 = (R + \rho) \cos \phi$, $x_2 = (R + \rho) \sin \phi$ in (31), to differentiate with respect to R, and to substitute into (39). The result is (34).

Equation 34 gives the effect of a slightly sloping, slightly warped M discontinuity when the crust is otherwise horizontally stratified and all materials are isotropic. Now consider the effect of slight horizontal variations in the properties of an isotropic crust and ocean, the M discontinuity being a horizontal plane and the uppermost mantle being isotropic and homogeneous. The properties of the ocean and crust are assumed to be described by (32). When α_i and β_{ij} are zero, the ray (path of stationary time) which descends from P to the M discontinuity, travels along the M discontinuity, and then ascends to Q will lie in a vertical plane. This ray we call the 'unperturbed' ray. When $v_M^{-1}L \langle|\alpha_i|\rangle$ and $v_M^{-1}L^2 \langle|\beta_{ij}|\rangle$ are small but not zero, the path of stationary time differs slightly from the unperturbed ray; we call it the 'perturbed' ray. If α_i and β_{ij} were zero, the travel times along the perturbed and unperturbed rays would be the same, to first order in the perturbation. Therefore the actual travel time along the perturbed ray can be obtained correctly to first order in $v_M^{-1}L \langle|\alpha_i|\rangle$ and $v_M^{-1}L^2 \langle|\beta_{ij}|\rangle$ by integrating (32) along the unperturbed ray. Differentiating the result with respect to distance from the fixed anchor point P to the variable point Q gives (36), with coefficients defined in (37) and (38).

Note that if slope and curvature of the M discontinuity, horizontal velocity variations, and anisotropy are all present but small, their cumulative effect is obtained by adding the corrections for their three separate effects obtained from (19), (34), and (36).

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