Navier-Stokesovy rovnice v aproximaci mělké vody

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Trocha historie...

Rovnice „mělké vody“

- horizontalní rozměry jsou mnohem větší než vertikální
- poprvé odvodil Adhémar Jean Claude Barré de Saint-Venant v roce 1871
- Théorie du mouvement non-permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leur lit. *C. R. Acad. Sc. Paris*, 73, 147-153.

Obrázek: A.J.C.B. de Saint-Venant (1797-1886)

Základní rovnice

- Navier-Stokesovy rovnice:

$$\nabla \cdot \vec{v} = 0$$

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \nabla \cdot (\vec{v} \otimes \vec{v}) \right) = \nabla \cdot \mathbf{t} + \vec{f}$$

- $\vec{f} = 2\vec{\Omega} \times \vec{v} + \vec{g}$
- $\mathbf{t} = -p\mathbf{I} + \boldsymbol{\sigma}$

Jak vypadá $\boldsymbol{\sigma}$

- Pro klasickou Newtonovskou kapalinu platí: $\boldsymbol{\sigma} = \eta(\nabla \mathbf{v} + \nabla^{\mathrm{T}} \mathbf{v})$.
- V případě proudění v oceánech je ale takový člen zanedbatelný.
- Poměr mezi nelineárním a viskózním členem udává Reynoldsovo číslo: $R_e = UL\rho/\eta$
- Typické hodnoty pro oceán: $U \approx 10^{-1}\mathrm{m/s}$, $L \approx 10^{6}\mathrm{m}$, $\rho \approx 10^{3}\mathrm{kg/m^{3}}$ a $\eta \approx 10^{-3}\mathrm{Pa.s} \quad \Rightarrow \quad R_e \approx 10^{11}$
- Hraniční hodnota mezi laminárním a turbulentním prouděním je zhruba $R_e \approx 1000$.
- Proudění v oceánech je tedy turbulentní a člen s molekulární viskozitu zanedbatelný, vnitřní tření kapaliny ale zcela zanedbat nelze!

Vliv turbulencí

- Proudění se skládá z časově zprůměrované složky, reprezentující velké škály a turbulentní složky na malých škálách

$$\vec{v} = \langle \vec{v} \rangle + \vec{v'}.$$

- Platí $\langle \vec{v'} \rangle = 0, \langle \langle \vec{v} \rangle + \vec{v'} \rangle = \langle \vec{v} \rangle$.
- Dosadíme-li do N-S rovnic a zprůměrujeme, pak nelineární člen v x-ová složce vypadá takto:

$$\left(\frac{\partial \langle u \rangle^2}{\partial x} + \frac{\partial (\langle u \rangle \langle v \rangle)}{\partial y} + \frac{\partial (\langle u \rangle \langle w \rangle)}{\partial z} \right) + \left(\frac{\partial \langle u' u' \rangle}{\partial x} + \frac{\partial \langle u' v' \rangle}{\partial y} + \frac{\partial \langle u' w' \rangle}{\partial z} \right).$$

Boussinesqova hypotéza

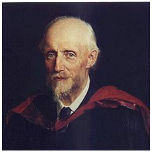
- Reynoldsovo napětí

$$\langle u'v' \rangle \equiv -\frac{\sigma_{xy}}{\rho}.$$

- Boussinesqova hypotéza

$$\frac{\sigma_{ij}}{\rho} = A_j \frac{\partial v_i}{\partial x_j} + A_i \frac{\partial v_j}{\partial x_i}.$$

- A_i je tzv. turbulentní viskozita.



Obrázek: Osborne Reynolds (1842-1912)

Obrázek: Joseph Valentin Boussinesq (1842-1929)

Reynoldsův tenzor a turbulentní viskozita

- Zavádí se Reynoldsův tenzor:

$$\frac{\boldsymbol{\sigma}}{\rho} = \begin{pmatrix} 2A_H \frac{\partial u}{\partial x} & A_H \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & A_V \frac{\partial u}{\partial z} + A_H \frac{\partial w}{\partial x} \\ A_H \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 2A_H \frac{\partial v}{\partial y} & A_V \frac{\partial v}{\partial z} + A_H \frac{\partial w}{\partial x} \\ A_V \frac{\partial u}{\partial z} + A_H \frac{\partial w}{\partial x} & A_V \frac{\partial v}{\partial z} + A_H \frac{\partial w}{\partial x} & 2A_V \frac{\partial w}{\partial z} \end{pmatrix}.$$

- A_H a A_V nejsou stejné a nejsou vlastností kapaliny, ale proudění
- Hodnoty pro oceánské proudění jsou
  - $A_H = 10 - 10^5 \mathrm{m}^2/\mathrm{s}$
  - $A_V = 10^{-5} - 10^{-1} \mathrm{m}^2/\mathrm{s}$
  - Pro srovnání kinematická viskozita vody $\nu = \eta/\rho \approx 10^{-6} \mathrm{m}^2/\mathrm{s}$.

Výchozí Navier-Stokesovy rovnice

$$\nabla \cdot \vec{v} = 0,$$

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv +$$

$$+ \frac{\partial}{\partial x}\left(2A_H \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H \frac{\partial u}{\partial y} + A_H \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(A_V \frac{\partial u}{\partial z} + A_H \frac{\partial w}{\partial x}\right),$$

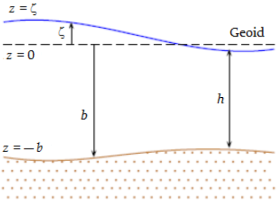
$$\frac{\partial v}{\partial t} + \frac{\partial (vu)}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial (vw)}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu +$$

$$+ \frac{\partial}{\partial x}\left(A_H \frac{\partial u}{\partial y} + A_H \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(2A_H \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_V \frac{\partial v}{\partial z} + A_H \frac{\partial w}{\partial y}\right),$$

$$\frac{\partial w}{\partial t} + \frac{\partial (wu)}{\partial x} + \frac{\partial (wv)}{\partial y} + \frac{\partial w^2}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g +$$

$$+ \frac{\partial}{\partial x}\left(A_V \frac{\partial u}{\partial z} + A_H \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_V \frac{\partial v}{\partial z} + A_H \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(2A_V \frac{\partial w}{\partial z}\right).$$

Okrajové podmínky



Na hladině:

$$w = \frac{\partial \zeta}{\partial t} + \vec{v} \cdot \nabla \zeta,$$
$$\boldsymbol{\sigma} \cdot \vec{n_s} = \vec{0}.$$

kde vnější normála je definována

$$\vec{n_s} = \frac{1}{\sqrt{1 + |\nabla \zeta|^2}} \begin{pmatrix} -\frac{\partial \zeta}{\partial x} \\ -\frac{\partial \zeta}{\partial y} \\ 1 \end{pmatrix}.$$

Okrajové podmínky

- Na dně:

$$w = -\vec{v} \cdot \nabla b,$$
$$\frac{1}{\rho}\boldsymbol{\sigma} \cdot \vec{n_b} + \vec{s_{fr}} = \vec{0},$$

s vnější normálou

$$\vec{n_b} = \frac{1}{\sqrt{1 + |\nabla b|^2}} \begin{pmatrix} -\frac{\partial b}{\partial x} \\ -\frac{\partial b}{\partial y} \\ -1 \end{pmatrix}.$$

- Tření na dně má kvadratický tvar: $\vec{s_{fr}} = k\vec{v}|\vec{v}|$.
- Na pobřeží: $\vec{v} \cdot \vec{n_c} = 0$

Bezrozměrný systém

- Zavedu charakteristické škály
  - L pro horizontální rozměr
  - H pro vertikální rozměr
  - U pro horizontání rychlost
- Dále zavedu:
  - „Malý“ parameter $\epsilon = H/L$
  - Inverzní Reynoldsova čísla $\nu_H = A_H/UL$ a $\nu_V = A_V/UL$, Froudovo číslo $F_r = U/\sqrt{gH}$ a Rossbyho číslo $R_o = U/fL$.
  - Odvozené škály pro vertikální rychlost $W = \epsilon U$, čas $T = L/U$ a tlak/hustota $P = U^2$.
- Přejdu do bezrozměrných veličin vydělením fyzikální veličiny příslušnou charakteristickou škálou.

Bezrozměrný systém

- Rovnice kontinuity zůstává stejná

$$\nabla \cdot \vec{v} = 0.$$

- Stejně tak zůstávají nezměněny okrajové podmínky na rychlost

$$w = \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} \qquad \text{pro } z = \zeta,$$

$$w = -u\frac{\partial b}{\partial x} - v\frac{\partial b}{\partial y} \qquad \text{pro } z = -b.$$

Bezrozměrný Navier-Stokes

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial (uv)}{\partial y}+\frac{\partial (uw)}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{v}{R_o}+$$
$$+\frac{\partial}{\partial x}\left(2\nu_H\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu_H\frac{\partial u}{\partial y}+\nu_H\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\nu_V}{\epsilon^2}\frac{\partial u}{\partial z}+\nu_H\frac{\partial w}{\partial x}\right),$$
$$\frac{\partial v}{\partial t}+\frac{\partial (vu)}{\partial x}+\frac{\partial v^2}{\partial y}+\frac{\partial (vw)}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial y}=-\frac{u}{R_o}+$$
$$+\frac{\partial}{\partial x}\left(\nu_H\frac{\partial u}{\partial y}+\nu_H\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(2\nu_H\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\nu_V}{\epsilon^2}\frac{\partial v}{\partial z}+\nu_H\frac{\partial w}{\partial y}\right),$$
$$\epsilon^2\left(\frac{\partial w}{\partial t}+\frac{\partial (wu)}{\partial x}+\frac{\partial (wv)}{\partial y}+\frac{\partial w^2}{\partial z}\right)+\frac{1}{\rho}\frac{\partial p}{\partial z}=-\frac{1}{F_r}+$$
$$+\frac{\partial}{\partial x}\left(\nu_V\frac{\partial u}{\partial z}+\epsilon^2\nu_H\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu_V\frac{\partial v}{\partial z}+\epsilon^2\nu_H\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial z}\left(2\nu_V\frac{\partial w}{\partial z}\right).$$

Bezrozměrný Navier-Stokes

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{v}{R_o} +$$
$$+ \frac{\partial}{\partial x}\left(2\nu_H \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu_H \frac{\partial u}{\partial y} + \nu_H \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\nu_V}{\epsilon^2}\frac{\partial u}{\partial z} + \nu_H \frac{\partial w}{\partial x}\right),$$
$$\frac{\partial v}{\partial t} + \frac{\partial (vu)}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial (vw)}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial y} = -\frac{u}{R_o} +$$
$$+ \frac{\partial}{\partial x}\left(\nu_H \frac{\partial u}{\partial y} + \nu_H \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(2\nu_H \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\nu_V}{\epsilon^2}\frac{\partial v}{\partial z} + \nu_H \frac{\partial w}{\partial y}\right),$$
$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{1}{F_r}.$$

Napěťová podmínka na povrchu

$$
\begin{aligned}
-\frac{\partial\zeta}{\partial x}\left(-\frac{p}{\rho}+2\nu_H\frac{\partial u}{\partial x}\right)-\frac{\partial\zeta}{\partial y}\nu_H\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\nu_V}{\epsilon^2}\frac{\partial u}{\partial z}+\nu_H\frac{\partial w}{\partial x}&=0,\\
-\frac{\partial\zeta}{\partial x}\nu_H\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-\frac{\partial\zeta}{\partial y}\left(-\frac{p}{\rho}+2\nu_H\frac{\partial v}{\partial y}\right)+\frac{\nu_V}{\epsilon^2}\frac{\partial v}{\partial z}+\nu_H\frac{\partial w}{\partial y}&=0,\\
-\epsilon^2\left(\frac{\partial\zeta}{\partial x}\left(\nu_H\frac{\partial w}{\partial x}+\frac{\nu_V}{\epsilon^2}\frac{\partial u}{\partial z}\right)+\frac{\partial\zeta}{\partial y}\left(\nu_H\frac{\partial w}{\partial y}+\frac{\nu_V}{\epsilon^2}\frac{\partial v}{\partial z}\right)\right)+2\nu_V\frac{\partial w}{\partial z}-\frac{p}{\rho}&=0.
\end{aligned}
$$

Napěťová podmínka na povrchu

$$-\frac{\partial\zeta}{\partial x}\left(-\frac{p}{\rho}+2\nu_H\frac{\partial u}{\partial x}\right)-\frac{\partial\zeta}{\partial y}\nu_H\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\nu_V}{\epsilon^2}\frac{\partial u}{\partial z}+\nu_H\frac{\partial w}{\partial x}=0,$$

$$-\frac{\partial\zeta}{\partial x}\nu_H\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-\frac{\partial\zeta}{\partial y}\left(-\frac{p}{\rho}+2\nu_H\frac{\partial v}{\partial y}\right)+\frac{\nu_V}{\epsilon^2}\frac{\partial v}{\partial z}+\nu_H\frac{\partial w}{\partial y}=0,$$

$$-\frac{p}{\rho}=0.$$

Napěťová podmínka na dně

$$\frac{\partial b}{\partial x}\left(-\frac{p}{\rho}+2\nu_H\frac{\partial u}{\partial x}\right)+\frac{\partial b}{\partial y}\nu_H\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\nu_V}{\epsilon^2}\frac{\partial u}{\partial z}+\nu_H\frac{\partial w}{\partial x}=Nk_0u|\vec{v}|,$$

$$\frac{\partial b}{\partial x}\nu_H\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\partial b}{\partial y}\left(-\frac{p}{\rho}+2\nu_H\frac{\partial v}{\partial y}\right)+\frac{\nu_V}{\epsilon^2}\frac{\partial v}{\partial z}+\nu_H\frac{\partial w}{\partial y}=Nk_0v|\vec{v}|,$$

$$\epsilon^2\left(\frac{\partial b}{\partial x}\left(\nu_H\frac{\partial w}{\partial x}+\frac{\nu_V}{\epsilon^2}\frac{\partial u}{\partial z}\right)+\frac{\partial b}{\partial y}\left(\nu_H\frac{\partial w}{\partial y}+\frac{\nu_V}{\epsilon^2}\frac{\partial v}{\partial z}\right)\right)+2\nu_V\frac{\partial w}{\partial z}-\frac{p}{\rho}=$$

$$=\epsilon^2Nk_0w|\vec{v}|,$$

$$N=\sqrt{1+\epsilon^2|\nabla b|^2},$$

$$|\vec{v}|=\sqrt{u^2+v^2+\epsilon^2w^2}.$$

Napěťová podmínka na dně

$$\frac{\partial b}{\partial x}\left(-\frac{p}{\rho}+2\nu_H\frac{\partial u}{\partial x}\right)+\frac{\partial b}{\partial y}\nu_H\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\nu_V}{\epsilon^2}\frac{\partial u}{\partial z}+\nu_H\frac{\partial w}{\partial x}=Nk_0u|\vec{v}|,$$

$$\frac{\partial b}{\partial x}\nu_H\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\partial b}{\partial y}\left(-\frac{p}{\rho}+2\nu_H\frac{\partial v}{\partial y}\right)+\frac{\nu_V}{\epsilon^2}\frac{\partial v}{\partial z}+\nu_H\frac{\partial w}{\partial y}=Nk_0v|\vec{v}|,$$

$$-\frac{p}{\rho}=0,$$

$$N=1,$$

$$|\vec{v}|=\sqrt{u^2+v^2}.$$

Vertikální zprůměrování

- NS rovnice přeintegrujeme podle z od $-b$ do ζ.
- Využijeme Leibnizovo integrální pravidlo

$$\int_{-b(\alpha)}^{\zeta(\alpha)} \frac{\partial}{\partial \alpha} f(z,\alpha) \mathrm{d}z = \frac{\partial}{\partial \alpha} \int_{-b(\alpha)}^{\zeta(\alpha)} f(z,\alpha) \mathrm{d}z - \frac{\partial \zeta}{\partial \alpha} f(\zeta,\alpha) - \frac{\partial b}{\partial \alpha} f(b,\alpha).$$

- Aplikujeme okrajové podmínky.
- Definujeme vertikálně zprůměrované veličiny

$$\bar{f}(t,x,y) = \frac{1}{h(t,x,y)} \int_{-b}^{\zeta} f(t,x,y,z) \, \mathrm{d}z.$$

Přeintegrované rovnice

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (h\bar{u})}{\partial x} + \frac{\partial (h\bar{v})}{\partial y} = 0,$$

$$\frac{\partial (h\bar{u})}{\partial t} + \frac{\partial (h\overline{u^2})}{\partial x} + \frac{\partial (h\overline{uv})}{\partial y} = -\frac{1}{\rho}\frac{\partial (h\bar{p})}{\partial x} - k_0 u\sqrt{u^2+v^2}|_{z=-b} + \frac{h\bar{v}}{R_o} + \\ + 2\nu_H \frac{\partial}{\partial x}\left(h\overline{\frac{\partial u}{\partial x}}\right) + \nu_H \frac{\partial}{\partial y}\left(h\overline{\frac{\partial u}{\partial y}} + h\overline{\frac{\partial v}{\partial x}}\right),$$

$$\frac{\partial (h\bar{v})}{\partial t} + \frac{\partial (h\overline{uv})}{\partial x} + \frac{\partial (h\overline{v^2})}{\partial y} = -\frac{1}{\rho}\frac{\partial (h\bar{p})}{\partial y} - k_0 v\sqrt{u^2+v^2}|_{z=-b} - \frac{h\bar{u}}{R_o} + \\ + \nu_H \frac{\partial}{\partial x}\left(h\overline{\frac{\partial u}{\partial y}} + h\overline{\frac{\partial v}{\partial x}}\right) + 2\nu_H \frac{\partial}{\partial y}\left(h\overline{\frac{\partial v}{\partial y}}\right).$$

$$\bar{p}(t,x,y) = \frac{\rho h}{2F_r}.$$

Slabá závislost na z

- Předpokládáme, že závislost horizontálních rychlostí na z je slabá

$$\frac{\partial u}{\partial z}(t,x,y,z) = O(\epsilon), \qquad \frac{\partial v}{\partial z}(t,x,y,z) = O(\epsilon).$$

- Taylorův rozvoj:

$$u(t,x,y,z) = \bar{u}(t,x,y) + O(\epsilon), \qquad v(t,x,y,z) = \bar{v}(t,x,y) + O(\epsilon).$$

- Což implikuje

$$\overline{u^2} = \bar{u}^2 + O(\epsilon), \qquad \overline{v^2} = \bar{v}^2 + O(\epsilon), \qquad \overline{uv} = \bar{u}\bar{v} + O(\epsilon),$$

$$\overline{\frac{\partial u}{\partial y}} = \frac{\partial \bar{u}}{\partial y} + O(\epsilon), \qquad \overline{\frac{\partial v}{\partial x}} = \frac{\partial \bar{v}}{\partial x} + O(\epsilon), \qquad \overline{\frac{\partial u}{\partial y}} + \overline{\frac{\partial v}{\partial x}} = \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}.$$

Výsledné rovnice

Výsledné bezrozměrné rovnice v lokálních kartézských souřadnicích v přesnosti $O(\epsilon)$

$$
\begin{aligned}
\frac{\partial \zeta}{\partial t} + \frac{\partial (h\bar{u})}{\partial x} + \frac{\partial (h\bar{v})}{\partial y} &= 0, \\
\frac{\partial (h\bar{u})}{\partial t} + \frac{\partial (h\bar{u}^2)}{\partial x} + \frac{\partial (h\bar{u}\bar{v})}{\partial y} &= -\frac{h}{F_r^2}\frac{\partial h}{\partial x} - k_0\bar{u}\sqrt{\bar{u}^2+\bar{v}^2} + \frac{h\bar{v}}{R_o} + \\
&\quad + 2\nu_H \frac{\partial}{\partial x}\left(h\frac{\partial \bar{u}}{\partial x}\right) + \nu_H \frac{\partial}{\partial y}\left(h\frac{\partial \bar{u}}{\partial y} + h\frac{\partial \bar{v}}{\partial x}\right), \\
\frac{\partial (h\bar{v})}{\partial t} + \frac{\partial (h\bar{u}\bar{v})}{\partial x} + \frac{\partial (h\bar{v}^2)}{\partial y} &= -\frac{h}{F_r^2}\frac{\partial h}{\partial y} - k_0\bar{v}\sqrt{\bar{u}^2+\bar{v}^2} - \frac{h\bar{u}}{R_o} + \\
&\quad + \nu_H \frac{\partial}{\partial x}\left(h\frac{\partial \bar{u}}{\partial y} + h\frac{\partial \bar{v}}{\partial x}\right) + 2\nu_H \frac{\partial}{\partial y}\left(h\frac{\partial \bar{v}}{\partial y}\right).
\end{aligned}
$$

Geografické souřadnice

- Nechť (λ_0, ϕ_0) určují počátek lokálního kartézského souřadného systému. Pak

$$
\begin{aligned}
x &= a\cos(\phi_0)(\lambda - \lambda_0), \\
y &= a(\phi - \phi_0).
\end{aligned}
$$

- A pro operátory derivace platí

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{1}{a\cos\phi}\frac{\partial}{\partial \lambda}, \\
\frac{\partial}{\partial y} &= \frac{1}{a}\frac{\partial}{\partial \phi}.
\end{aligned}
$$

- Navíc si označím $U = h\bar{u}$ a $V = h\bar{v}$.

Finální rovnice

Finální rovnice „mělké vody“ v geografických rovnicích:

$$\frac{\partial h}{\partial t}+\frac{1}{a\cos\phi}\frac{\partial U}{\partial \lambda}+\frac{1}{a}\frac{\partial V}{\partial \phi}=0,$$

$$\frac{\partial U}{\partial t}=-\frac{1}{a\cos\phi}\frac{\partial U\bar{u}}{\partial \lambda}-\frac{1}{a}\frac{\partial U\bar{v}}{\partial \phi}-\frac{gh}{a\cos\phi}\frac{\partial h}{\partial \lambda}-\frac{k}{h^2}U\sqrt{U^2+V^2}+fV+$$
$$+\frac{2A_H}{a^2\cos^2\phi}\frac{\partial}{\partial \lambda}\left(h\frac{\partial \bar{u}}{\partial \lambda}\right)+\frac{A_H}{a^2}\frac{\partial}{\partial \phi}\left(h\frac{\partial \bar{u}}{\partial \phi}+\frac{h}{\cos\phi}\frac{\partial \bar{v}}{\partial \lambda}\right),$$

$$\frac{\partial V}{\partial t}=-\frac{1}{a\cos\phi}\frac{\partial V\bar{u}}{\partial \lambda}-\frac{1}{a}\frac{\partial V\bar{v}}{\partial \phi}-\frac{gh}{a}\frac{\partial h}{\partial \phi}-\frac{k}{h^2}V\sqrt{U^2+V^2}-fU+$$
$$+\frac{A_H}{a^2\cos\phi}\frac{\partial}{\partial \lambda}\left(h\frac{\partial \bar{u}}{\partial \phi}+\frac{h}{\cos\phi}\frac{\partial \bar{v}}{\partial \lambda}\right)+\frac{2A_H}{a^2}\frac{\partial}{\partial \phi}\left(h\frac{\partial \bar{v}}{\partial \phi}\right).$$