Influence of material parameters on stability of thermal convection

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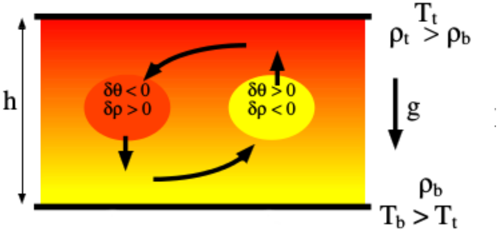
Introduction

Rayleigh-Bénard Problem, critical Rayleigh number

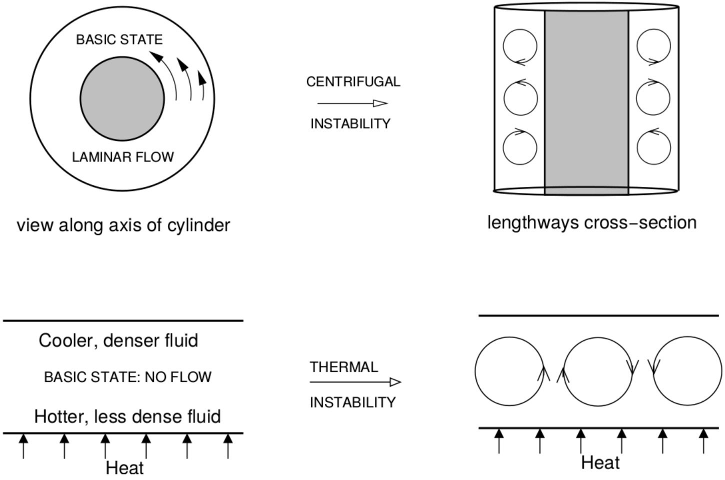


Material parameters:

- viscosity: $\mu = \mu(p, T)$
- thermal expansivity: $\alpha = \alpha(p, T)$
- thermal conductivity: $k = k(p, T)$

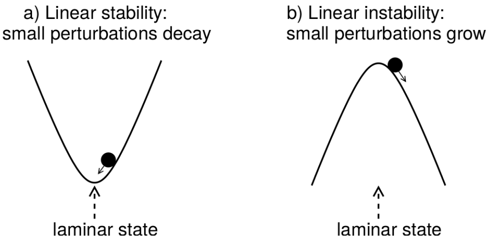
Dissipation, heat sources...

Lineary stability analysis



Lineary stability analysis

- **System of nonlinear equations governing the problem**: Navier-Stokes-Fourier equations (with a specified parameter λ), boundary conditions
- **Basic flow** (time-independent solutions): $\mathbf{v}_B(\mathbf{x})$, $p_B(\mathbf{x})$, $T_B(\mathbf{x})$
- **Perturbations from the basic flow**: $\mathbf{v}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$, $T(\mathbf{x}, t)$



Lineary stability analysis

Basic flow is said to be

- **stable** (in the sense of Ljapunov) if $\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon)$: $\|\clubsuit(\mathbf{x}, 0)\| < \delta \Rightarrow \|\clubsuit(\mathbf{x}, t)\| < \varepsilon \ \forall t \geq 0$.
- **asymptotically stable** if moreover $\|\clubsuit(\mathbf{x}, t)\| \to 0$ as $t \to \infty$.

Here $\clubsuit$ denotes perturbations $\mathbf{v}$, p or T.

Linear stability analysis

Questions:

- (Q1) For any value of the control parameter λ, is the basic laminar state lineary stable or unstable, i.e. do the perturbations decay or grow in time?
- (Q2) What is the threshold value of λ at which the laminar state first becomes unstable?
- (Q3) At the onset of instability, what is the spatial form of the unstable perturbations, and how fast do they grow?

Linear stability analysis

Linearized problem, normal modes analysis

- We put $T_B(\mathbf{x}, t) + T(\mathbf{x}, t)$ (basic flow + perturbation) into the governing equations (N-S-F) and neglect all nonlinear terms
- We assume the perturbations to have the form

$$T(\mathbf{x}, t) = \operatorname{Re} \sum_{k,l=-\infty}^{\infty} \int \tilde{T}(z, k, l, \sigma) e^{i(k a_x x + l a_y y) + \sigma t} \mathrm{d}\sigma$$

All together, we arrive at a generalized eigenvalue problem

$$\mathcal{L}\{(\mathbf{v}, p, T); \sigma, \lambda\} = \mathbf{0}.$$

Linear (in)stability of the basic flow:

- $\operatorname{Re} \sigma \leq 0 \Rightarrow$ mode is (asymptotically) stable
- $\operatorname{Re} \sigma > 0 \Rightarrow$ mode is unstable

Balance equations

Continuity equation

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{v}) = 0,$$

Equation of motion

$$\varrho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} \right] = \varrho \mathbf{g} - \nabla p + \nabla\left(\lambda \operatorname{div} \mathbf{v}\right) + \operatorname{div}\left(2\mu \mathbb{D}\right),$$

Heat equation

$$\varrho c_p \frac{\partial T}{\partial t} = \operatorname{div}(k \nabla T) - \varrho c_p \mathbf{v} \cdot \nabla T - \varrho v^{\hat{z}} \alpha T g + \lambda (\operatorname{div} \mathbf{v})^2 + 2\mu \mathbb{D} : \mathbb{D} + Q,$$

+ Equation of state

$$\varrho = \varrho(p, T).$$

Boussinesq approximation

Linearization of the equation of state (neglect density changes caused by the pressure deviations):

$$\varrho = \varrho_0 \left[1 - \alpha(T - T_0)\right].$$

Anelastic liquid approximation:

$$\mathrm{div}(\varrho_0 \mathbf{v}) = 0,$$

$$\varrho_0 \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v}\right] = -\varrho_0 \alpha (T - T_0)\mathbf{g}_0 - \nabla \Pi + \nabla(\lambda\, \mathrm{div}\, \mathbf{v}) + \mathrm{div}(2\mu \mathbb{D}),$$

$$\varrho_0 c_p \frac{\partial T}{\partial t} = \mathrm{div}(k \nabla T) - \varrho_0 c_p \mathbf{v} \cdot \nabla T - \varrho_0 v^{\hat{r}} \alpha T g + \lambda(\mathrm{div}\, \mathbf{v})^2 + 2\mu \mathbb{D} : \mathbb{D} + Q.$$

Boussinesq approximation

Extended Boussinesq approximation:

$$\operatorname{div} \mathbf{v} = 0,$$

$$\varrho_0 \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} \right] = -\varrho_0 \alpha (T - T_0) \mathbf{g}_0 - \nabla \Pi + \operatorname{div}(2\mu \mathbb{D}),$$

$$\varrho_0 c_p \frac{\partial T}{\partial t} = \operatorname{div}(k \nabla T) - \varrho_0 c_p \mathbf{v} \cdot \nabla T - \varrho_0 v^{\hat{r}} \alpha T g + 2\mu \mathbb{D} : \mathbb{D} + Q.$$

Classical Boussinesq approximation ($\varrho_0, \mathbf{g}_0, \alpha, c_p, k, \mu = \text{const}$):

$$\operatorname{div} \mathbf{v} = 0,$$

$$\varrho_0 \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} \right] = -\varrho_0 \alpha (T - T_0) \mathbf{g}_0 - \nabla \Pi + \mu \Delta \mathbf{v},$$

$$\frac{\partial T}{\partial t} = \kappa \Delta T - \mathbf{v} \cdot \nabla T,$$

Rayleigh-Bénard problem

Balance equations in dimensionless form, Cartesian geometry $\rightarrow$

$$\mathrm{div}\,\mathbf{v} = 0,$$

$$\frac{1}{Pr}\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right] = -Ra(T - T_0)\mathbf{e}_{\hat{z}} - \nabla\Pi + \Delta\mathbf{v},$$

$$\frac{\partial T}{\partial t} = \Delta T - \mathbf{v}\cdot\nabla T.$$

Prandtl number $\qquad Pr = \dfrac{\mu}{\varrho_0 \kappa}$

Rayleigh number $\qquad Ra = \dfrac{\varrho_0 \alpha (T_b - T_s) g_0 d^3}{\mu\kappa}$

Rayleigh-Bénard problem

Basic flow: $\mathbf{v} = \mathbf{0}$, $T_0 = T_0(z)$, $\Pi = 0$.

$$\frac{\partial T}{\partial t} = \Delta T - \mathbf{v} \cdot \nabla T \quad \Rightarrow \quad \frac{\mathrm{d}^2 T_0}{\mathrm{d}z^2} = 0 \quad \Rightarrow \quad T_0 = z.$$

Perturbations of the basic flow:

$$\mathbf{v}' = \mathbf{0} + \mathbf{v} = \mathbf{v}, \quad T' = T_0 + \theta, \quad \Pi' = 0 + \pi = \pi.$$

Linear stability analysis:

$$\operatorname{div} \mathbf{v} = 0,$$

$$\frac{1}{Pr}\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right] = -Ra\theta \mathbf{e}_{\hat{z}} - \nabla \pi + \Delta \mathbf{v},$$

$$\frac{\partial \theta}{\partial t} = \Delta \theta - v^{\hat{z}} \frac{\mathrm{d}T_0}{\mathrm{d}z} - \mathbf{v} \cdot \nabla \theta.$$

Rayleigh-Bénard problem

Perturbations in the form of normal modes:

$$\mathbf{v}(t,\mathbf{x}) \propto \tilde{\mathbf{v}}(z)\exp\left[i(a_x x + a_y y) + \sigma t\right].$$

$$\theta(t,\mathbf{x}) \propto \tilde{\theta}(z)\exp\left[i(a_x x + a_y y) + \sigma t\right].$$

Generalized eigenvalue problem:

$$\sigma \mathbb{A}\begin{bmatrix}\tilde{v}^{\hat{z}}\\ \tilde{\theta}\end{bmatrix} = \mathbb{B}\begin{bmatrix}\tilde{v}^{\hat{z}}\\ \tilde{\theta}\end{bmatrix},$$

where

$$\mathbb{A} = \begin{bmatrix}\frac{1}{Pr}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - a^2\right) & 0\\ 0 & 1\end{bmatrix}, \quad \mathbb{B} = \begin{bmatrix}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - a^2\right)^2 & a^2 Ra\\ -1 & \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - a^2\right)\end{bmatrix}.$$

Chebyshev spectral method

1950s: finite difference methods
1960s: finite element methods
1970s: **spectral methods** (high accuracy, demand less computer memory)

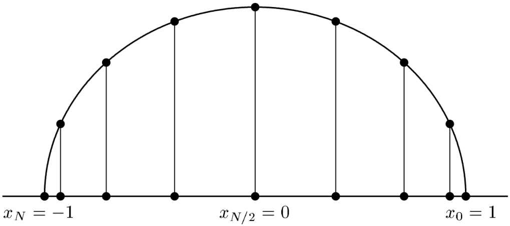


Figure : Clustered grid (Gauss-Chebyshev-Lobatto points)

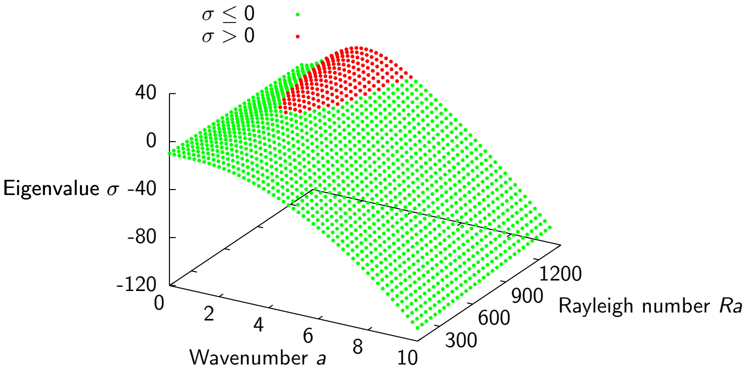
Chebyshev difference matrices → algebraic eigenvalue problem → solved using `Matlab`

Rayleigh-Bénard problem

- **stress-free**:

$$\tilde{v}^{\hat{z}}\Big|_{z=0,1} = \frac{\mathrm{d}^2\tilde{v}^{\hat{z}}}{\mathrm{d}z^2}\Big|_{z=0,1} = 0, \quad \tilde{\theta}\Big|_{z=0,1} = 0$$

$$\boxed{Ra_{\mathrm{crit}} \approx 657.55, \quad a_{\mathrm{crit}} \approx 2.222}$$

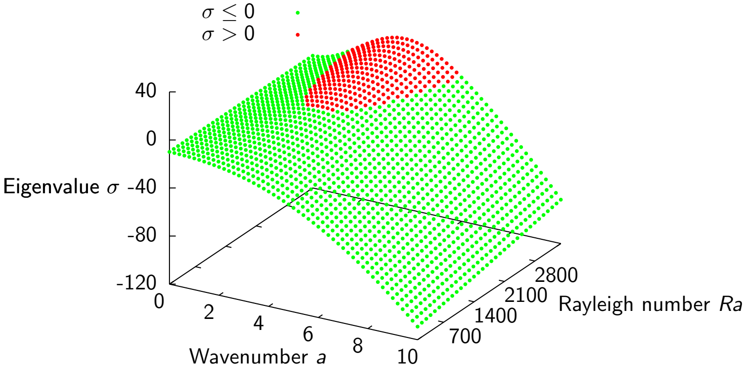


Rayleigh-Bénard problem

- **no-slip**:

$$\tilde{v}^{\hat{z}}\Big|_{z=0,1} = \frac{\mathrm{d}\tilde{v}^{\hat{z}}}{\mathrm{d}z}\Big|_{z=0,1} = 0, \quad \tilde{\theta}\Big|_{z=0,1} = 0$$

$$\boxed{Ra_{\mathrm{crit}} \approx 1707.83, \quad a_{\mathrm{crit}} \approx 3.116}$$



Rayleigh-Bénard problem

Marginal curves

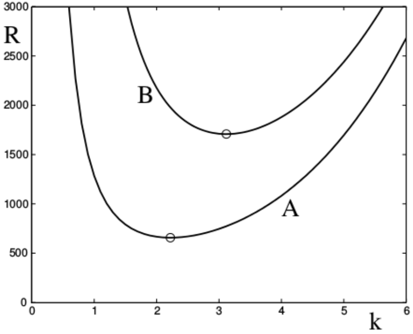


Fig. 3. Marginal stability curves for stress-free (A) and no-slip (B) velocity boundary conditions and isothermal plates (infinitely large conductivity).

Extended Boussinesq problem

Balance equations in dimensionless form, Cartesian geometry $\rightarrow$

$$\operatorname{div}\mathbf{v} = 0,$$

$$\frac{1}{Pr_s}\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right] = -Ra_s\alpha(T - T_0)\mathbf{e}_{\hat{z}} \\ - \nabla\Pi + \operatorname{div}\left(\mu\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right]\right),$$

$$\frac{\partial T}{\partial t} = \operatorname{div}(k\nabla T) - \mathbf{v}\cdot\nabla T + Di_s\alpha\left(T + \frac{Ra_s^*}{Ra_s}\right)v^{\hat{z}} \\ + \frac{Di_s}{Ra_s}\mu\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right] : \nabla\mathbf{v} + \frac{Raq_s}{Ra_s}.$$

Extended Boussinesq problem

Dimensionless numbers:

Prandtl number	$Pr_s = \dfrac{\mu_s}{\varrho_0 \kappa_s}$
Rayleigh number	$Ra_s = \dfrac{\varrho_0 \alpha_s (T_b - T_s) g_0 d^3}{\mu_s \kappa_s}$
Rayleigh number for heat sources	$Raq_s = \dfrac{\varrho_0 \alpha_s g_0 Q d^5}{\mu_s \kappa_s k_s}$
"Rayleigh number for T_s"	$Ra_s^* = \dfrac{\varrho_0 \alpha_s T_s g_0 d^3}{\mu_s \kappa_s}$
dissipation number	$Di_s = \dfrac{\alpha_s g_0 d}{c_p}.$

Material parameters:

thermal conductivity	$k = k(z, T)$
thermal expansivity	$\alpha = \alpha(z, T)$
viscosity	$\mu = \mu(z, T)$

Properties of the Earth's mantle

Dimensionless numbers:[1]
$Pr \approx 10^{22}$ (infinite Prandtl number approximation may be used)
$Di_s = 0.5$
$Raq_s = 3 \times 10^7$ (however, we will be using $Raq_s = 10^6$)
$Ra_s^* = 8.1 \times 10^5$

Material coefficients

$$
\begin{aligned}
k(z, T) &= k_s + b_k z + c_k (T - T_s), \\
\alpha(z, T) &= \alpha_s \exp(-b_\alpha z - c_\alpha (T - T_s)), \\
\mu(z, T) &= \mu_s + b_\mu z + c_\mu (T - T_s),
\end{aligned}
$$

We will also consider **stress-free** boundary conditions only.

[1] Matyska, C. and Yuen, D.A.: Lower-mantle material properties and convection models of multiscale plumes

Reference temperature distribution

Basic flow: $\mathbf{v} = \mathbf{0}$, $T_0 = T_0(z)$, $\Pi = 0$.

Heat equation reduces to

$$0 = \operatorname{div}\left(k(z, T_0)\nabla T_0\right) + \frac{Raq_s}{Ra_s}.$$

Second order non-linear differential equation - solved again using spectral collocation method.

Generalized eigenvalue problem

- Perturbations of the basic flow **v**, θ and π
- Linearization of the equations
- Normal modes analysis
- Generalized eigenvalue problem:

$$\sigma \mathbb{A} \begin{bmatrix} \tilde{v}^{\hat{z}} \\ \tilde{\theta} \end{bmatrix} = \mathbb{B} \begin{bmatrix} \tilde{v}^{\hat{z}} \\ \tilde{\theta} \end{bmatrix}.$$

$$\left(\quad B_{21} = -\frac{\mathrm{d}T_0}{\mathrm{d}z} + Di_s \alpha(z, T_0) \left(T_0 + \frac{Ra_s^*}{Ra_s} \right) \quad \right)$$

Critical Rayleigh numbers, extended Boussinesq

stress-free boundary conditions
depth-dependent thermal conductivity: $k(z, T) = 1 + \tilde{b}_k z$

$\tilde{b}_k$	Di_s	Raq_s	a_{crit}	Ra_{crit}
1	0	0	2.221	976.933
5	0	0	2.220	2 210.40
10	0	0	2.218	3 767.83
1	0.5	0	6.334	404 688
5	0.5	0	7.086	344 453
10	0.5	0	7.155	342 840
1	0.5	10^6	7.565	225 133
5	0.5	10^6	7.612	274 150
10	0.5	10^6	7.465	303 949

Critical Rayleigh numbers, extended Boussinesq

stress-free boundary conditions
temperature-dependent viscosity $\mu(z, T) = 1 + \tilde{c}_k Ra_s T$

$\tilde{c}_\mu$	Di_s	Raq_s	a_{crit}	Ra_{crit}
1×10^{-7}	0	0	2.221	657.532
10×10^{-7}	0	0	2.221	657.728
100×10^{-7}	0	0	2.222	657.679
1×10^{-7}	0.5	0	5.356	475 809
10×10^{-7}	0.5	0	5.305	477 023
100×10^{-7}	0.5	0	4.875	486 968
1×10^{-7}	0.5	10^6	7.452	178 666
10×10^{-7}	0.5	10^6	7.391	182 395
100×10^{-7}	0.5	10^6	6.837	214 301

Comparison with the classical case

Classical Boussinesq approximation:

$$\boxed{Ra_{\mathrm{crit}} \approx 657.51, \quad a_{\mathrm{crit}} \approx 2.222}$$

Extended Boussinesq approximation for $Di_s = 0.5$, $Raq_s = 3 \times 10^7$ and all the material coefficients being functions of depth and temperature:

$$\boxed{Ra_{\mathrm{crit}} \approx 556\,740, \quad a_{\mathrm{crit}} \approx 4.986}$$

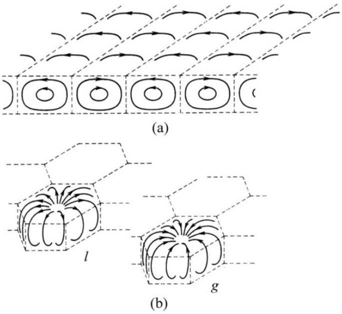
Real value of Ra_s in Earth's mantle $\approx 10^7$.

What happens next?

Recap:

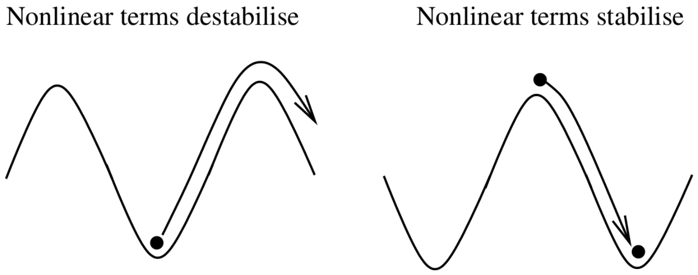
- Ra_{crit}, a_{crit}
- eigenfunctions $\tilde{v}^{\hat{z}}$, $\tilde{\theta}$
- horizontal solutions undetermined ($e^{i(a_x x + a_y y)}$, $a_x^2 + a_y^2 = a_{\mathrm{crit}}^2$)

Many possible solutions (rolls, hexagons, squares, rectangles, parallelograms):



Weakly nonlinear analysis

For $Ra > Ra_{\mathrm{crit}}$ $\exists$ eigenvalue σ s.t. $\mathrm{Re}\,\sigma > 0 \Rightarrow$ exponential growth of the perturbation in time. At later times, once the perturbation has grown to attain a larger amplitude, the assumption of linearity breaks down. Nonlinear effects become significant.



Weakly nonlinear analysis

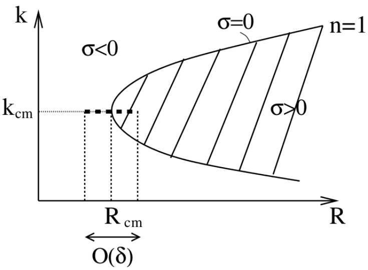
The linear theory gives the most unstable mode with

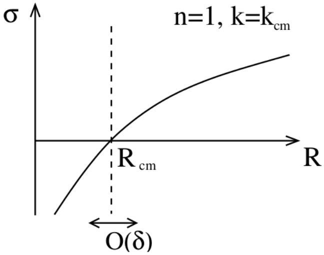
$$\theta(\mathbf{x}, t) = A(t) f(x, y, a) \tilde{\theta}_1(z)$$

where

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \sigma A.$$

We could Taylor expand σ in the vicinity of the critical pair $(Ra_{\mathrm{crit}}, a_{\mathrm{crit}})$





Stuart-Landau equation

The amplitude A of the most unstable mode is in fact governed by the so called **Stuart-Landau equation** (also called amplitude equation)

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} = \sigma A + \beta A|A|^2.}$$

We will show how to derive this equation on a simple model example.

First bifurcation

Dynamical behaviour predicted by Stuart-Landau equation:
$A(t) = \rho(t)e^{i\phi(t)} \Rightarrow$

$$\dot{\rho} = \mathrm{Re}(\sigma)\rho + \mathrm{Re}(\beta)\rho^3$$

$$\dot{\theta} = \mathrm{Im}(\sigma) + \mathrm{Im}(\beta)\rho^2$$

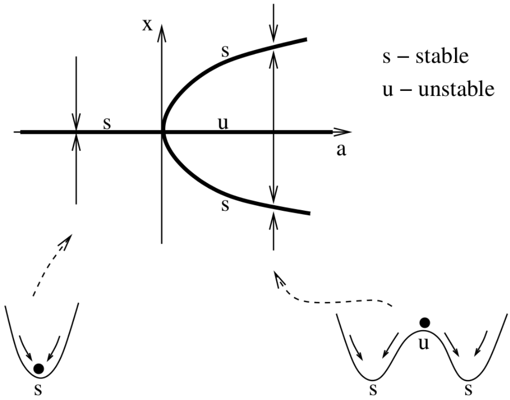
Two types of bifurcations can occur depending on whether σ_c, β are complex or not:

- σ, $\beta \in \mathbb{R}$: supercritical/subcritical pitchfork bifurcation
- $\sigma \in \mathbb{C}$: supercritical/subcritical Hopf bifurcation

In our case: $\sigma \in \mathbb{R}$, $\beta < 0$. That means *supercritical pitchfork bifurcation*.

First bifurcation

Pitchfork bifurcation



Amplitude equation

Consider

$$\frac{\partial u}{\partial t} = \frac{1}{\lambda}\frac{\partial^2 u}{\partial x^2} + u + u\frac{\partial u}{\partial x}, \qquad u(0) = u(1) = 0,$$

where $\lambda \in \mathbb{R}$ is a control parameter.
Base state: $u_B \equiv 0$.
Linear stability analysis:

$$\sigma u = \frac{1}{\lambda}u'' + u,$$

yields

$$\sigma_n = 1 - \frac{n^2\pi^2}{\lambda}, \qquad u_n(x) = \sqrt{2}\sin(n\pi x).$$

The linear operator is self-adjoint, hence σ_n are distinct and simple and $\{u_n\}_{n=1}^{+\infty}$ forms an orthonormal basis. This allows the expansion

$$u(t,x) = \sum_{n=1}^{+\infty} A_n(t)u_n(x).$$

Amplitude equation

Subtituting the expansion into the original non-linear equation yields

$$\sum_{n=1}^{+\infty} \frac{\mathrm{d}A_n}{\mathrm{d}t} u_n = \sum_{n=1}^{+\infty} \sigma_n A_n u_n + \sum_{n=1}^{+\infty} \sum_{n=1}^{+\infty} A_n A_m u_n \frac{\mathrm{d}u_m}{\mathrm{d}x},$$

where we used the fact that $\{u_n\}_{n=1}^{+\infty}$ are the eigenfunctions of the linear operator.

Projection on the eigenfunctions:

$$\frac{\mathrm{d}A_k}{\mathrm{d}t} = \sigma_k A_k + \sum_{n=1}^{+\infty} \sum_{n=1}^{+\infty} A_n A_m \langle u_n \frac{\mathrm{d}u_m}{\mathrm{d}x}, u_k \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $L^2([0,1])$.

Amplitude equation

The unstable mode ($k = 1$):

$$\frac{\mathrm{d}A_1}{\mathrm{d}t} \approx \sigma_1 A_1 + \sum_{k=1}^{+\infty} A_1 A_k \left(\langle u_k \frac{\mathrm{d}u_1}{\mathrm{d}x}, u_1 \rangle + \langle u_1 \frac{\mathrm{d}u_k}{\mathrm{d}x}, u_1 \rangle \right).$$

The decaying modes ($k \geq 2$):

$$0 = \sigma_k A_k + \sum_{n=1}^{+\infty} \sum_{n=1}^{+\infty} A_n A_m \langle u_n \frac{\mathrm{d}u_m}{\mathrm{d}x}, u_k \rangle$$

$$A_k = -\frac{1}{\sigma_k} \sum_{n=1}^{+\infty} \sum_{n=1}^{+\infty} A_n A_m \langle u_n \frac{\mathrm{d}u_m}{\mathrm{d}x}, u_k \rangle \approx -\frac{1}{\sigma_k} A_1^2 \langle u_1 \frac{\mathrm{d}u_1}{\mathrm{d}x}, u_k \rangle$$

Amplitude equation

The amplitude equation

$$\frac{\mathrm{d}A_1}{\mathrm{d}t} = \sigma_1 A_1 - \sum_{k=1}^{+\infty} A_1^3 \frac{1}{\sigma_k} \langle u_1 \frac{\mathrm{d}u_1}{\mathrm{d}x}, u_k \rangle \left(\langle u_k \frac{\mathrm{d}u_1}{\mathrm{d}x}, u_1 \rangle + \langle u_1 \frac{\mathrm{d}u_k}{\mathrm{d}x}, u_1 \rangle \right).$$

In fact the infinite series is non-zero for $k = 2$ only. Hence we arrive at

$$\boxed{\frac{\mathrm{d}A_1}{\mathrm{d}t} = \sigma_1 A_1 + \frac{\pi^2}{16\sigma_2} A_1^3}$$

Note: We can apply this in the interval $[\pi^2, 4\pi^2)$ only.

Amplitude equation for thermal convection

The system of equations governing the perturbations for either classical or extended Boussinesq approximation can be written as

$$\frac{\partial \boldsymbol{\psi}}{\partial t} = \mathcal{L}(Ra)(\boldsymbol{\psi}) + \mathcal{N}(\boldsymbol{\psi}, \boldsymbol{\psi}),$$

where

$$\boldsymbol{\psi} = \begin{bmatrix} \theta & v^{\hat{x}} & v^{\hat{z}} \end{bmatrix}^T,$$

$\mathcal{L}$ is a linear operator and $\mathcal{N}$ is a quadratic non-linear operator.

The linear operator $\mathcal{L}$ for extended Boussinesq approximation lacks the important property of being **self-adjoint** (or at least normal).Two possibilities are at hand:

- orthogonalize the eigenfunctions
- use the expansion in terms of non-orthogonal eigenfunctions

Amplitude equation: numerical results

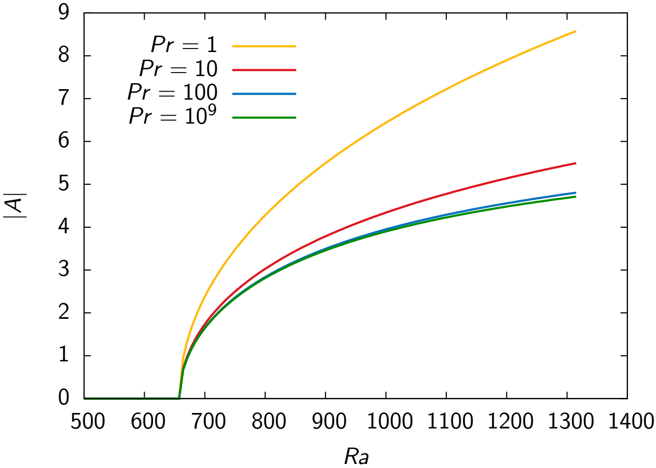


Figure : Classical Boussinesq approximation, stress-free boundary condition

Amplitude equation: numerical results

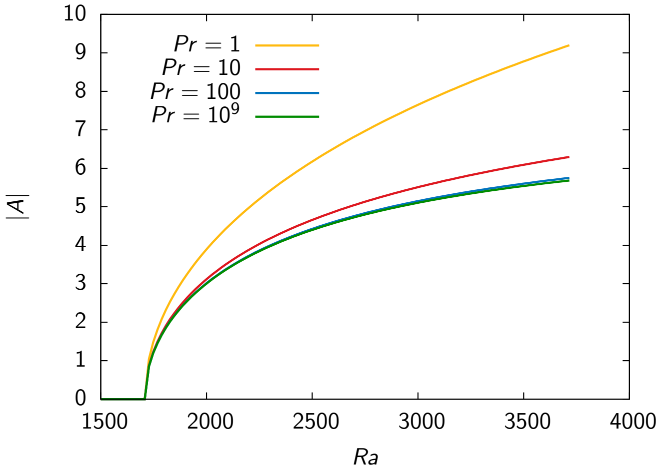


Figure : Classical Boussinesq approximation, no-slip boundary condition

Amplitude equation: numerical results

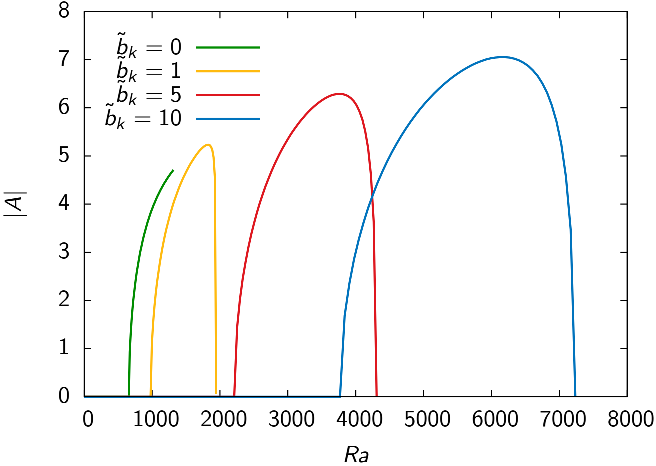


Figure : Extended Boussinesq approximation, stress-free boundary condition, $k(z, T) = 1 + \tilde{b}_k z$, $Pr_s = 10^9$, $Di_s = 0$, $Raq_s = 0$

Amplitude equation: numerical results

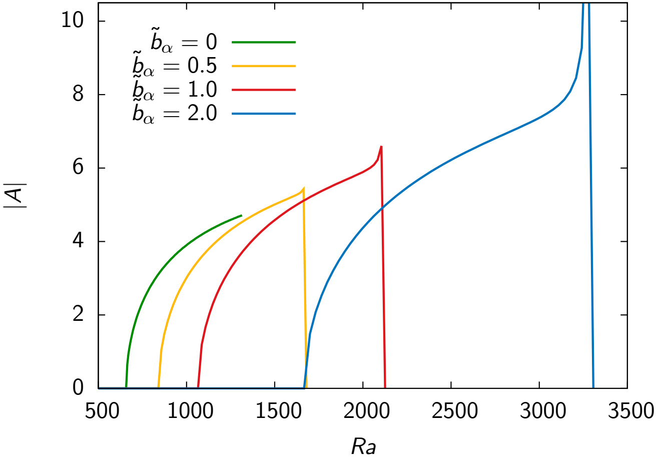


Figure : Extended Boussinesq approximation, stress-free boundary condition, $\alpha(z, T) = \exp(-\tilde{b}_\alpha z)$, $Pr_s = 10^9$, $Di_s = 0$, $Raq_s = 0$

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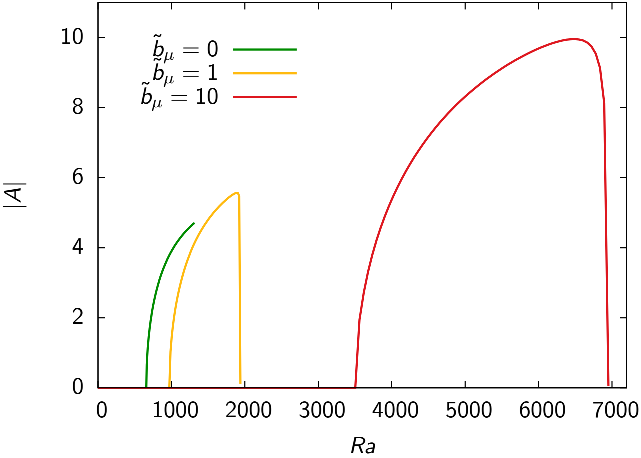

Figure : Extended Boussinesq approximation, stress-free boundary condition, $\mu(z, T) = 1 + \tilde{b}_\mu z$, $Pr_s = 10^9$, $Di_s = 0$, $Raq_s = 0$

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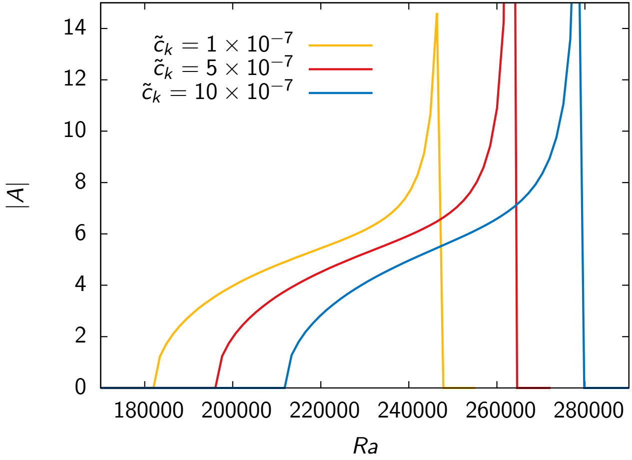

Figure : Extended Boussinesq approximation, stress-free boundary condition, $k(z, T) = 1 + \tilde{c}_k Ra_s T$, $Pr_s = 10^9$, $Di_s = 0.5$, $Raq_s = 10^6$

Conclusions

We have found critical Rayleigh numbers for the onset of convection in various cases of non-constant material coefficients k, α and μ.
We have also investigated the character of convection for slightly super-critical Rayleigh numbers.
Drawbacks:

- The method works only for smooth enough data.
- The method works only for large enough Prandtl numbers (approximately 10^3 and larger).
- There are many more bifurcations as the Rayleigh number rises. We investigated only the first one...

Thank you for your attention.