3. The “Sea Level Equation” - theory and applications
(November 9, start 13:10)

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November 7–11, 2011
Outline

1). Surface loads ("ice" and "water loads")

2). Response to surface loads

3). “The Sea Level Equation” (with discussion)

4). Special topics: global RSL variations, a SLE benchmark, and present-day geodetic variations.

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Surface loads:

- ice and water
- but also volcanoes...
- time-dependent,
- globally distributed,
- often unknown a priori,
- and interacting,
...
A general load can be decomposed as:

\[ L(\omega, t) = L_i + L_o \]

with:

\[ \omega = (\theta, \lambda) \quad t = \text{time}, \]

and:

\[ L_i = \text{ice load} \]
\[ L_o = \text{ocean (water) load} \]

that describe changes in the weight of the ice sheets and the oceans, respectively:

\[ L_i \] describes “Glacial isostasy”
\[ L_o \] describes “Hydro isostasy”
Ice loads
Ice loads are related to ice thickness variations, NOT to absolute thickness (*):

\[ T(\omega, t) \]

\[ I(\omega, t) = T(\omega, t) - T(\omega, t_0) \]

The "ice load" is defined as:

\[ L_i(\omega, t) = \frac{\text{Ice mass variation}}{\text{Unit surface}} = \rho_i I(\omega, t) \]

Hence the variation of ice sheets mass is:

\[ m_i(t) = a^2 \int_i d\omega L_i(\omega, t) \]

(*) Note: the SLE does not include any "ABSOLUTE" quantity, only variations...
If the ice load has *fixed margins*, the load can be “factorized”:

\[ L_i(\omega, t) = \sigma(\omega) f(t) \]

where:

\[ \sigma(\omega) \quad \text{“load function” a closed form often exists} \]

\[ f(t) \quad \text{“load time-history” e. g., in the form of steps…} \]

Generally the load time history has a step-like form:

\[ f(t) = f_0 + \sum_{i}^{N} (f_{i+1} - f_i)[H(t - t_{i+1}) - H(t - t_i)] \]

The case of a complex ice load with *time-evolving margins* can be dealt with by a combination of factorized loads of this type.
Northern hemisphere glaciation during the last ice ages. The set up of 3 to 4 km thick ice sheets caused a sea level lowering of about 120 m.
Ice loads examples of ice loads

“disc load”

“disc load”

“disc load”

“disc loads”

“quadrilateral loads”

ICE3G (N)

ICE1 (N)
Ice loads

The thickness of each disc varies in a “piecewise manner” (Isostatic equilibrium *may be* assumed before the LGM)

![Diagram](image)

- $H_0$
- $H_1$
- $H_2$
- $H_k$

LGM to today
The full ICE1

Authors: Peltier and Andrews, GJRAS, 1976.

153 number of elements
20 header lines
1 lines for each element
18 time since the LGM (kyrs)
1 length of each time step
5 size of the 'rectangular' elements (deg)

For each ice element, the data are presented in one line, with the following meaning:
colatitude and longitude of the centroid of the ('rectangular') elements (deg), and
disc element thickness (m) at various times (as integers). The first value is
thickness before the beginning of deglaciation (i.e. from \(-\infty\) to 18 kyrs BP),
the 18 elements that follow are ice thickness at 1 kyrs intervals up to present time.
File SH1.F of SELEN reads this file and indicates with \(h_0\) ice thickness
before melting begins, \(h_1\) between 18 and 17 kyrs BP, ..., and with \(h_{18}\) ice thickness
between 1 and 0 kyrs BP. With \(h_{19}=h_{18}\) SELEN denotes the ice (constant) thickness in

10 265 500 300 300 300 300 200 200 200 200 0 0 0 0 0 0 0 0 0
10 270 1000 850 850 850 850 800 800 800 800 900 900 500 500 500 500 500 500 500 500 500
10 275 1000 1250 1250 1250 1250 1200 1200 1200 1200 1200 1200 1200 1000 1000 500 500 500 500 500 500 500 500 500
10 280 2000 1250 1250 1250 1250 1200 1200 1200 1200 1200 1200 1200 1000 1000 500 500 500 500 500 500 500 500 500 500
10 285 1000 650 650 650 650 600 600 600 600 600 600 600 600 200 200 500 500 500 500 500 500 500 500
15 250 200 300 300 300 300 300 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
15 255 200 1000 1000 1000 1000 1000 1000 1000 1000 1000 1000 1000 1000 1000 1000 0 0 0 0 0 0 0 0
15 260 1000 1400 1400 1400 1400 1400 1400 1400 1400 1400 1400 1400 1400 1400 1400 0 0 0 0 0 0 0 0
15 265 1000 1500 1500 1500 1500 1500 1500 1500 1500 1500 1500 1500 1500 1500 1500 600 600 0 0 0 0 0 0 0 0 0
15 270 1000 1200 1200 1200 1200 1200 1200 1200 900 900 700 700 0 0 0 0 0 0 0 0 0 0 0 0 0 0
15 275 1000 900 900 900 900 800 800 800 800 800 800 800 800 800 900 900 500 500 500 500 500 500 500 500 500 500

Friday, November 11, 2011
Ice loads examples of ice loads - ICE5G

ICE5G thickness until 21 kyrs BP

ICE5G thickness (m)
Total ESL = 127.40 m

ICE5G ESL (m)

ESL = \frac{\rho_i V}{\rho_w A_o}

CRE's ?

LGM

today

Friday, November 11, 2011
Ice loads

examples: ICE-1 and ICE-3G

ICE1(LGM)

Peltier & Andrews, 1976
ESL = 80m
Based on glaciology

ICE1(0 BP)

Peltier & Andrews, 1976
ESL = 80m
Based on glaciology
Ice loads examples: ICE-1 and ICE-3G

ICE3G (LGM)

Tushingham & Peltier, 1992
ESL = 120 m
Recently updated to ICE5G
Based on glaciology and constrained by RSL data

ICE3G (0 BP)

Tushingham & Peltier, 1992
ESL = 120 m
Recently updated to ICE5G
Based on glaciology and constrained by RSL data
(melt)water load
Before introducing the “water load”, we define important quantities:

- **Sea Level change**: $SL = r_g - r_t$
- **Sea surface height variation**: $N$
- **Vertical displacement**: $U$

**Illustration of Sea Level**

Sea Level change = geoid height variation - vertical displacement

**“SEA LEVEL EQUATION” (SLE):**

$$S = N - U$$
The “water load” is defined as:

\[ L_o(\omega, t) = \frac{\text{Oceans mass variation}}{\text{Unit surface}} \]

or equivalently:

\[ L_o(\omega, t) = \rho_w I(\omega, t) O(\omega, t) \]

with:

- \( \rho_w \) water density
- \( S(\omega, t) \) sea level change
- \( O(\omega, t) \) (time-dependent) ocean function

\[ O(\omega, t) = \begin{cases} 
1, & \omega \in \text{oceans} \\
0, & \omega \notin \text{oceans} 
\end{cases} \]
The water load can be estimated assuming:

1) a rigid Earth: Love numbers = 0, and

2) no gravitational interactions: Newton constant = 0.

In this case, the sea level variations are “EUSTATIC”, from “eustatism” a word coined by the English geologist Eduard Suess (1903)

\[ S = \text{constant} \]
The law of eustasy can be stated as follows:

\[ S(\omega, t) = -\frac{\text{Ice mass variation}}{\text{Water density} \times \text{Area of the oceans}} \]

In other words:

\[ S = S^E(t) = -\frac{m_i(t)}{\rho_w A_o} \quad \text{Sea level change is everywhere the SAME} \]

\[ L_o^E(\omega, t) = -\frac{m_i(t)}{A_o} \mathcal{O}(\omega, t) \quad \text{Eustatic water load is constant over the oceans} \]

\[ m_i(t) = M_i(t) - M_i(t_r) \quad \text{Ice mass variation} \]

\[ M_i(t) \quad \text{Effective ice mass} \]
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2). Response to surface loads

3). “The Sea Level Equation” (with discussion)

4). Special topics: global RSL variations, a SLE benchmark, and present-day geodetic variations.
The Green function can be used to compute the response of the Earth to FINITE-SIZED loads (this is the reason of its usefulness!).

The vertical displacement $U$ and the variation of the gravitational potential $\Phi$ at any point $\omega$ and time $t$ result from changes of both ice and water mass distributions at any other point $\omega'$ and times $t' \leq t$.

Mathematically, this can be stated by convolution integrals over the surface of the Earth:

$$\begin{bmatrix} U \\ \Phi \end{bmatrix}(\omega, t) = \int_{-\infty}^{t} dt' \int_{e} dA' \begin{bmatrix} G_u \\ G_\phi \end{bmatrix}(\alpha, t - t') \mathcal{L}(\omega', t')$$

where $\mathcal{L}(\omega, t)$ is TOTAL surface load and $dA = a^2 \sin \theta d\theta d\lambda$ is the element of area on the surface of the sphere.
Response to surface loads

We introduce the simplified notation:

\[(G \otimes_D L)(\omega, t) \equiv \int_{-\infty}^{t} dt' \int_D dA' G(\alpha, t - t') L(\omega', t')\]

where \(D\) is a subset of the surface of the Earth, \(L(\omega, t)\) is a generic surface load, and \(G\) is a given Green function.

Thus:

\[
\left\{ \begin{array}{c} U \\ \Phi \end{array} \right\} (\omega, t) = \left\{ \begin{array}{c} G_u \\ G_\phi \end{array} \right\} \otimes_e \mathcal{L} = \left\{ \begin{array}{c} U_i + U_o \\ \Phi_i + \Phi_o \end{array} \right\}
\]

where:

\[
\left\{ \begin{array}{c} U_i \\ \Phi_i \end{array} \right\} (\omega, t) \equiv \left\{ \begin{array}{c} G_u \\ G_\phi \end{array} \right\} \otimes_i \rho_i I, \quad \text{and} \quad \left\{ \begin{array}{c} U_o \\ \Phi_o \end{array} \right\} (\omega, t) \equiv \left\{ \begin{array}{c} G_u \\ G_\phi \end{array} \right\} \otimes_o \rho_w S
\]

are the “ice” and “ocean components” of \(U\) and \(\Phi\), respectively.
Response to surface loads

“self-gravitation”

An observation. In the case of RIGID EARTH,

\[ G_u = 0 \quad \text{but:} \quad G_\phi \neq 0 \]

because of the “1” term (the “DIRECT” effect) in the expression for the GF for the gravity potential.

Hence, from:

\[
\begin{align*}
\left\{ \frac{U_i}{\Phi_i} \right\} (\omega, t) & \equiv \left\{ \frac{G_u}{G_\phi} \right\} \otimes_i \rho_i I, \quad \text{and} \quad \left\{ \frac{U_o}{\Phi_o} \right\} (\omega, t) & \equiv \left\{ \frac{G_u}{G_\phi} \right\} \otimes_o \rho_w S
\end{align*}
\]

we see that, for a RIGID EARTH: \( U_i = U_o = 0 \quad \text{but:} \quad \)

\( \Phi_i \neq 0 \quad \text{Gravitational interaction: ice sheets - oceans} \)

\( \Phi_o \neq 0 \quad \text{Gravitational interaction: oceans - oceans} \)

(we say that the oceans are “self-gravitating”).
Response to surface loads              a problem

Equations:

\[ \begin{align*}
\left\{ \begin{array}{l}
U_i \\
\Phi_i
\end{array} \right\}(\omega, t) & \equiv \left\{ \begin{array}{l}
G_u \\
G_\phi
\end{array} \right\} \otimes_i \rho_i I,
\text{ and } \left\{ \begin{array}{l}
U_o \\
\Phi_o
\end{array} \right\}(\omega, t) & \equiv \left\{ \begin{array}{l}
G_u \\
G_\phi
\end{array} \right\} \otimes_o \rho_w S
\end{align*} \]

pose a serious problem:

Solving e.g. for \( U_i \) and \( U_o \) requires knowledge of \( I \) and \( S \). While \( I \) may be "assumed" to be known a-priori (e.g., from geological evidence), \( S \) will be only known after vertical displacements (and geoid) are determined!

So, to compute \( S \), we need \( U \) (and the geoid), but to compute \( U \) (and the geoid), we need \( S \)!

How to escape from this circular argument?
There are two possible solutions!

The first is to assume a-priori eustatic sea-level variations.

This leads to what I call the “simplified GIA problem”, in which an eustatic water load is assumed (*). The simplified GIA problem has been dealt with in Lesson 2 ---> see the benchmark paper.

The major limitation of this approach is that it cannot accurately describe sea level variations in the far-field of the ice sheets, where water load is comparable (or in excess to) ice load.

In the near-field, the simplified GIA problem usually provides accurate solutions...

We can do better, but we must follow a second, more difficult route...

Is sea level change uniform across the oceans, far away from the (former) ice sheets?
RSL observations

Fig. 2. Observed spatial variability of sea level change since the time of the LGM from tectonically stable areas or areas where the tectonic rate is known and has been removed from the observed signal. (A) Ångermanland, Gulf of Bothnia, Sweden (13). (B) Andøya, Nordland, Norway (12). (C) South of England (14). (D) Hudson Bay, Canada (4). (E) Barbados (16–18). (F) Bonaparte Gulf, northwest Australia (27). (G) Orpheus Island, North Queensland, Australia (23) and unpublished Australian National University data. (H) Sunda Shelf, southeast Asia (15). Note the different time and amplitude scales. In the examples illustrated, all observed depths or elevations of the sea level indicators have been reduced to mean sea level. All time scales are in calendar years.

Sea Level Change Through the Last Glacial Cycle
Kurt Lambeck, et al.
Science 292, 679 (2001);
DOI: 10.1126/science.1059549
no, it is not.

We definitively need a “Gravitationally Self-Consistent” theory for Sea Level change!
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$$N = r_{\text{geoid}}(\omega, t) - r_{\text{geoid}}(\omega, t_0)$$

Sea surface AND equipotential

$$SL = r_{\text{geoid}}(\omega, t) - r_{\text{topo}}(\omega, t)$$

$$U = r_{\text{topo}}(\omega, t) - r_{\text{topo}}(\omega, t_0)$$

$$S = SL(\omega, t) - SL(\omega, t_0) = N - U$$

“Sea Level Equation”
The sea level equation (SLE) - basic ideas

ICE LOADING

A

B

C

GRAVITATIONAL ATTRACTION

D

E

F

MELT WATER LOADING

DELAYED RESPONSE

G

Clark & Lingle, 1979
The sea level equation (SLE) -

Write the sea surface variation as: \[ N = \frac{\Phi}{\gamma} + c \]

By the elementary form of the SLE: \[ S = \frac{\Phi}{\gamma} + c - U \]

Impose mass conservation: \[ c = -\frac{m_i}{\rho_w A_o} - \left( \frac{\Phi}{\gamma} - U \right) \]

Hence:
\[ S = \left( \frac{\Phi}{\gamma} - U \right) + S^E - \left( \frac{\Phi}{\gamma} - U \right) \]
The sea level equation (SLE) -

Using the “convolution” expressions for $\Phi$ and $U$ finally gives:

$$S = \frac{\rho_i}{\gamma} G_s \otimes_i I + \frac{\rho_w}{\gamma} G_s \otimes_o S - \frac{m_i}{\rho_w A_o} - \frac{\rho_i}{\gamma} G_s \otimes_i I - \frac{\rho_w}{\gamma} G_s \otimes_o S$$

with:

$S =$ sea level change

$\rho_i, \rho_w =$ ice and water density

$G_s =$ sea level Green function

$I =$ ice thickness variation

$m_i =$ ice mass variation

$A_o =$ area of the oceans

$\otimes_i, \otimes_o =$ 3(2+1)D convolutions

$(\ldots) =$ ocean average
The sea level equation (SLE) -

More details can be found in these papers:


A discussion about the SLE (in 10 points and remarks)

\[ S = \frac{\rho_i}{\gamma} G_s \otimes_i I + \frac{\rho_w}{\gamma} G_s \otimes_o S + S^E - \frac{\rho_i}{\gamma} G_s \overline{\otimes_i I} - \frac{\rho_w}{\gamma} G_s \overline{\otimes_o S} \]
1) **SL Green function.**
The SLE contains the “Sea Level Green function”, which combines the vertical displacement and gravity potential GFs:

\[ S = \frac{\rho_i}{\gamma} G_s \otimes i I + \frac{\rho_w}{\gamma} G_s \otimes_o S - \frac{m_i}{\rho_w A_o} - \frac{\rho_i}{\gamma} G_s \otimes_i I - \frac{\rho_w}{\gamma} G_s \otimes_o S \]

\[ G_s(\alpha, t) \equiv \frac{G_\phi}{\gamma} - G_u \quad \text{Depends upon “h” and “k”, not on “l”} \]

Horizontal movements are *caused* by sea level change S, but they *do not influence* S.

Their importance has been recognized since the 90’s in de-glaciated areas. Rule of thumb: 10 times smaller that vertical motions...
2) **Geoid.**

“The” equipotential surface that represents the free surface of the oceans.

The SLE ensures that the time-evolving geoid is always an equipotential surface.

3) **Mass conservation.**

The SLE accounts automatically for *mass conservation* within the system composed by (oceans + ice sheets + solid Earth). The atmosphere is ignored in GIA modeling...

In GIA computations, density of ice and water are assumed to be constant. *Regardless of temperature variations of the oceans, dynamics of the ice sheets, etc...*

*Regarding ice: ice is a “boundary condition”. No physical coupling with bedrock, except for the effect of weight.*
A discussion about the SLE -

More observations on the SLE...

\[ S = \frac{\rho_i}{\gamma} G_s \otimes_i I + \frac{\rho_w}{\gamma} G_s \otimes_o S - \frac{m_i}{\rho_w A_o} - \frac{\rho_i}{\gamma} G_s \otimes_i I - \frac{\rho_w}{\gamma} G_s \otimes_o S \]

4) **Nature of the SLE.** The SLE is a **linear, integral equation**, since the **unknown S** also appears within convolution integrals!

This reflects the simple idea that sea level change at one point \( \omega \) and time \( t \) depends on sea level change at any point \( \omega' \) and at any time \( t' \leq t \)!

In other words... << the problem of sea level change can not be reduced to a regional scale >>.
More observations on the SLE...

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More observations on the SLE...

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In other words... << the problem of sea level change can not be reduced to a regional scale >>.
More observations...

5) **Eustasy.** The SLE has a very special solution when several terms are dropped:

\[
S = \frac{\rho_i}{\gamma} G_s \otimes_i I + \frac{\rho_w}{\gamma} G_s \otimes_o S - \frac{m_i}{\rho_w A_o} - \frac{\rho_i}{\gamma} G_s \otimes_i I - \frac{\rho_w}{\gamma} G_s \otimes_o S
\]

We are left with:

\[
S \equiv S^E = - \frac{m_i}{\rho_w A_o}
\]

that represents the “eustatic” solution of Suess.

For a RIGID, non-gravitating Earth (Green functions = 0), the SLE has the eustatic solution. The solution is only determined by the mass variation of the ice sheet and the area of the oceans!
A discussion about the SLE -

More observations...

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A discussion about the SLE -

More observations...

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For a RIGID, non-gravitating Earth (Green functions = 0), the SLE has the eustatic solution. The solution is only determined by the mass variation of the ice sheet and the area of the oceans!
5) **Eustasy.**

Because of glacial fluctuations true, or eustatic, sea level (which those of us who live along oceanic coasts may think of as a constant, ignoring waves, tides, and storms) varies across spans of observation ranging from decades to hundreds of thousands of years. This variation, in turn, is superimposed on sea level fluctuations across millions of years caused by changes in the volume of the ocean basins.

\[
S = S^E(t) = - \frac{m_i(t)}{\rho_w A_o}
\]

is solution of the SLE if:

\[
U = 0 \quad \text{and} \quad \Phi = 0
\]

that is, if:

\[
G_u = G_\phi = 0
\]

RIGID EARTH

+ NO GRAVITATION
7) More observations on Eustasy.

Since the real Earth is not rigid and Newton's constant is not zero, we must expect that the melting of the ice sheets actually produces spatially non-uniform sea level variations, which may significantly depart from eustasy.

However, the SLE has a peculiar property. We take the ocean-average of it:

$$
\overline{S} = \frac{\rho_i}{\gamma} G_s \otimes_i \overline{I} + \frac{\rho_w}{\gamma} G_s \otimes_o S + \overline{SE} - \frac{\rho_i}{\gamma} G_s \otimes_i \overline{I} - \frac{\rho_w}{\gamma} G_s \otimes_o \overline{S}
$$

Hence:

$$
\overline{S} = \overline{SE}
$$

In words: «<the ocean-averaged sea level change coincides with the sea level change we would observe in eustatic conditions!»
7) More observations on Eustasy.

if the ocean-averaged sea level change coincides with the sea level change we would observe in eustatic conditions, it follows that the history of the volume of ice sheets could be estimated by averaging the observed RSL curves over the oceans!

The RSL sites are not uniformly distributed over the oceans... does an average make sense?
8) **About the “Woodward solution”**.

Woodward (1888)* found an *analytical* solution of the SLE assuming that:

1- *the Earth is rigid*,

2- *the ice sheet is localized and impulsive* – so that here we face the *instantaneous freezing (or melting) of a “point load”*,

3- *the oceans cover uniformly the Earth, except from a point-like “continent”*,

4- *variations in the ocean mass distribution do not affect the gravity field* – *i. e.*, “*self–gravitation of the oceans*” is neglected.

Thus, the ONLY PHYSICAL EFFECT that is accounted for is gravitational attraction between the ICE and the OCEANS …

A discussion about the SLE - the SLE

“Woodward problem”

Frozen point mass

Micro-continent ("Lilliput")

Observer

Rigid Earth

Uniform ocean
A discussion about the SLE -

Seen the strong simplifying assumptions, the Woodward problem has a simple closed-form solution*:

\[ S^W(\alpha, t) = \frac{a}{me} \left( \frac{1}{2 \sin(\alpha/2)} - \frac{\bar{\rho}_e}{3\rho_w} - 1 \right) \]


\( S^W(\alpha, t) \) Sea level variation  \( me \) Mass of the Earth
\( \alpha \) Colatitude \( \bar{\rho}_e \) Average Earth density
\( t \) time \( \rho_w \) Water density

\( a \) Radius of the Earth
“Woodward problem”: Freezing of a point mass

- Sea level rise
- Sea level fall
- Divergence (effect of the point mass)
- No SL change here
- Eustatic SL change only here
- SL well below eustatic level

Sea level change (arbitrary units)

$\alpha$, colatitude from the point mass (degrees)

$S = 0$

$S = S^E$
9) **HOW to solve the SLE?** A few ideas here ...

The SLE "looks like" a "*inhomogeneous integral Fredholm equation of the 2nd type*", where the unknown occurs both inside and outside the integral, and the integration interval is fixed (fixed shorelines, in our language), which in one dimension has the form:

\[
\varphi(x) = f(x) + \lambda \int_{a}^{b} K(x, x') \varphi(x') dx',
\]

The problem can be solved by "iteration" of a first-guess solution:

0. Assume \( \varphi(x) = \varphi^{(0)} \)

1. This gives \( \varphi^{(1)}(x) = f(x) + \lambda \int_{a}^{b} K(x, x') \varphi^{(0)} dx' \)

... ... ...

n. At step "n": \( \varphi^{(n)}(x) = f(x) + \lambda \int_{a}^{b} K(x, x') \varphi^{(n-1)}(x') dx' \)

... ... ...

*continue until convergence* ...
9) **HOW to solve the SLE?** A few ideas here ...

\[ m_i(t) \]

Given the ice chronology, compute the ice mass variation

\[ S^E = -\frac{m_i}{\rho_w A_o} \]

Then, evaluate the eustatic sea level change

\[ S^0 = S^E \]

At step #0, sea level change is eustatic

\[ S^1 = \frac{\rho_i}{\gamma} G_s \otimes_i I + \frac{\rho_w}{\gamma} G_s \otimes_o S^0 + S^0 - \frac{\rho_i}{\gamma} G_s \otimes_i I - \frac{\rho_w}{\gamma} G_s \otimes_o S^0 \]

\[ \ldots \ldots \]

\[ S^k = \frac{\rho_i}{\gamma} G_s \otimes_i I + \frac{\rho_w}{\gamma} G_s \otimes_o S^{k-1} + S^0 - \frac{\rho_i}{\gamma} G_s \otimes_i I - \frac{\rho_w}{\gamma} G_s \otimes_o S^{k-1} \]

\[ \ldots \ldots \]

\[ \int_{\text{sphere}} \frac{|S^k - S^{k-1}|}{|S^k|} dA \leq \epsilon \]

Stopping criterion

(~5 iterations are sufficient)
Numerical approach:

- Pixelization by the Icosahedron method (Tegmark, ‘96)
- Pseudo-spectral approach,
- Ice models are discretized in space (grid + harmonics) and time (piecewise constant)
- Evolving shorelines and rotational feedbacks
- NEW: HPC applications with INGV Rome & ISCRA (CINECA).
10) On the meaning of some quantities relevant to the SLE:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Observed by</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>Sea Level Change</td>
<td>Tide gauges</td>
</tr>
<tr>
<td>$N$</td>
<td>&quot;Absolute&quot; sea surface variation</td>
<td>Altimetry</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>Gravity potential variation</td>
<td>Gravimetry (e.g. GRACE)</td>
</tr>
<tr>
<td>$U$</td>
<td>Vertical displacement</td>
<td>GPS</td>
</tr>
<tr>
<td>$V$</td>
<td>Horizontal displacement</td>
<td>GPS</td>
</tr>
<tr>
<td>$RSL$</td>
<td>Relative Sea Level</td>
<td>Geological record</td>
</tr>
</tbody>
</table>
10) Some interconnections between quantities relevant to the SLE

\[ S = N - U \]

Sea Level Change, from the solution of the SLE \((S)\)

\[ U = U(S, I) \]

From the ice history \((I)\) and the SLE \((S)\)

\[ \Phi = \Phi(S, I) \]

\[ V = V(S, I) \]

\[ G = \frac{\Phi}{\gamma} \]

By the SLE \((S)\) and mass conservation

\[ N = \frac{\Phi}{\gamma} + c \]

\[ c = -\frac{m_i}{\rho_w A_o} - \left( \frac{\Phi}{\gamma} - U \right) \]

Ensures mass conservation

\[ RSL \]

RSL derives from \(S >\) as we will see in one moment...
Outline

1). Surface loads ("ice" and "water loads")

2). Response to surface loads

3). "The Sea Level Equation" (with discussion)

4). Special topics: global RSL variations and a SLE benchmark, and maps of geodetic quantities.
Special topic (1/3)
Relative Sea Level (RSL)
What is a Relative Sea Level (RSL) curve? 
How do we construct it from the SLE? 
useful for interpretation of geological observations...
Relative Sea Level (RSL)  “Clark zones”

Clark & Lingle, 1979

Fig. 1. Distribution of six predicted sea level zones resulting from melting of ice sheets 20,000–7,000 yr ago. Within each zone the form of the sea level response is similar. Typical curves indicating the predicted change in sea level relative to the present geoid for each zone are included (RSL means relative sea level). The curves show the wide variety of sea level expressions possible despite the assumption of no change in ocean volume during the past 5,000 yr (no change in ‘eustatic’ sea level) [from Clark and Lingle, 1979].
Relative Sea Level (RSL) definitions

or, equivalently:

\[
S = \frac{\rho_i}{\gamma} G_s \otimes_i I + \frac{\rho_w}{\gamma} G_s \otimes_o S - \frac{m_i}{\rho_w A_o} - \frac{\rho_i}{\gamma} G_s \otimes_i I - \frac{\rho_w}{\gamma} G_s \otimes_o S
\]

with:

- \( S \) = sea level change
- \( m_i \) = ice mass variation
- \( \rho_i, \rho_w \) = ice and water density
- \( A_o \) = area of the oceans
- \( G_s \) = sea level Green function
- \( \otimes_i, \otimes_o \) = 3(2+1)D convolutions
- \( I \) = ice thickness variation
- \( \overbrace{(...)} = \) ocean average
Relative Sea Level (RSL)

\[ S = N - U \]

or, equivalently:

\[
S = \frac{\rho_i}{\gamma} G_s \otimes_i I + \frac{\rho_w}{\gamma} G_s \otimes_o S - \frac{m_i}{\rho_w A_o} - \frac{\rho_i}{\gamma} G_s \otimes_i \langle I \rangle - \frac{\rho_w}{\gamma} G_s \otimes_o \langle S \rangle
\]

with:

- \( S \) = sea level change
- \( \rho_i, \rho_w \) = ice and water density
- \( m_i \) = ice mass variation
- \( A_o \) = area of the oceans
- \( G_s \) = sea level Green function
- \( \otimes_i, \otimes_o \) = \(3(2+1)\)D convolutions
- \( I \) = ice thickness variation
- \( \langle \ldots \rangle \) = ocean average
The SLE does **not** provide absolute sea level variations, but rather it allows to compute “sea level change”. Solving the SLE gives “$S$”:

$$S(\omega, t_{BP}) = SL(\omega, t_{BP}) - SL(\omega, t_r)$$

with:

- $t_{BP}$ time Before Present (BP),
- $t_r$ a reference “remote” time BP,
- $SL$ Sea Level (sea surface-topography offset).

At present time, we can similarly write:

$$S(\omega, t_p) = SL(\omega, t_p) - SL(\omega, t_r)$$

$t_p$ present time.
The relative sea level (RSL) “relative” to a given epoch BP is **defined** as the past sea level referred to the present–day level:

\[ RSL(\omega, t_{BP}) = SL(\omega, t_{BP}) - SL(\omega, t_p) \]

that can be rephrased as follows:

\[ RSL(\omega, t_{BP}) = SL(\omega, t_{BP}) - SL(\omega, t_r) \]

\[ -(SL(\omega, t_p) - SL(\omega, t_r)) = \]

\[ = S(\omega, t_{BP}) - S(\omega, t_p) \]

which is directly related to the sea level change \( S \), i.e., the solution of the SLE.
What is the shape of the RSL curve for a RIGID, non-gravitating Earth?

In this case, the SLE has the “Eustatic” solution:

\[ S = S^E(t) = -\frac{m_i(t)}{\rho_w A_o} \]

This gives:

\[ S(t_{BP}) = -\frac{m_i(t_{BP})}{\rho_w A_o} = -\frac{M_i(t_{BP}) - M_i(t_r)}{\rho_w A_o} \]

\[ S(t_p) = -\frac{m_i(t_p)}{\rho_w A_o} = -\frac{M_i(t_p) - M_i(t_r)}{\rho_w A_o} \]

Hence, following the definition, RSL is:

\[ RSL(\omega, t_{BP}) = SL(\omega, t_{BP}) - SL(\omega, t_p) = -\frac{M_i(t_{BP}) - M_i(t_p)}{\rho_w A_o} \]
A case study:

“melting at a constant rate”.

The mass is constant until the “LGM” (Last Glacial Maximum).

After a phase with constant melting rate, the mass remains constant.

The RSL curve (blue, solid) can be derived by the mass variation curve (red, dashed) by reflection and rescaling.

The RSL curve is “Eustatic”, in this case.
Global Relative Sea Level Histories

Entry ID: EARTH_LAND_NGDC_PALEO_SEALEVEL1

Summary
The Relative Sea Level Database covers sea level data from the present time back to 14,000 years before the present. The authors compiled the database from numerous published sources in addition to their own field work. Methodology for establishing relative sea level histories varies between sites and investigators, but typically entails geomorphologic ...

Click to View Full Summary
Global RSL data

An example of HOLOCENE RSL global database: The Peltier dataset is hosted by NOAA (National Oceanographic and Atmospheric Administration).

Data Set Citation

Dataset Creator: Peltier, W.R. and A.M. Tushingham
Dataset Title: Relative Sea Level Database
Dataset Series Name: IGBP PAGES/World Data Center for Paleoclimatology Data Contribution
Dataset Release Date: 1993
Dataset Release Place: Boulder, CO
Dataset Publisher: NOAA/NCDC Paleoclimatology Program
Issue Identification: 93-016

Paleo Temporal Coverage
Paleo Start Date: 14,000 ybp
Paleo Stop Date: 0 ybp

http://gcmd.nasa.gov/records/GCMD_EARTH_LAND_NGDC_PALEO_SEALEVEL1.html
Relative Sea Level (RSL) sites
from file sealevel.dat -NRSL= 392

Sites in the RSL NOOA database
Summary
The Relative Sea Level Database covers sea level data from the present time back to 14,000 years before the present.

The authors compiled the database from numerous published sources in addition to their own field work.

Methodology for establishing relative sea level histories varies between sites and investigators, but typically entails geomorphologic methods to identify relict shoreline deposits, which are then dated radiometrically from associated carbonate or organic materials.

Final results are presented as relative height in meters versus modern mean sea level.

Publications/References

Original Reference:

The microfiche appendix of the Original Reference (Tushingham and Peltier, 1992) contains the complete bibliography, and is available from AGU, 2000 Florida Ave. NW, Washington, D.C., 20009 USA.


http://gcmd.nasa.gov/records/GCMD_EARTH_LAND_NGDC_PALEO_SEALEVEL1.html
The following files are available from the FTP server:

-----------------------------------------------------------------------

**rsl.readme** - readme file

**sealevel.** - 392 individual Relative Sea Level data files; There are 392 files named sealevel.*, each containing data from one site. These files are located in the subdirectory sitefiles.

**sealevel.dat** - All Relative Sea Level data files, concatenated; The file sealevel.dat contains radiocarbon-controlled relative sea level histories from 392 sites.

**rslsite.lst** - list of sites and file names in rsl database; A list of site names and locations can be found in the file rslsite.lst.

**rslform.lst** - FORTRAN code used to write sealevel.* data files

[http://gcmd.nasa.gov/records/GCMD_EARTH_LAND_NGDC_PALEO_SEALEVEL1.html](http://gcmd.nasa.gov/records/GCMD_EARTH_LAND_NGDC_PALEO_SEALEVEL1.html)
Global RSL data

**a Relative Sea Level database**

http://www1.ncdc.noaa.gov/pub/data/paleo/paleocean/relative\sea\level/
Global RSL data

**Relative Sea Level**

*a Relative Sea Level database*

http://www1.ncdc.noaa.gov/pub/data/paleo/paleocean/relative\sea\level/
a Relative Sea Level database
http://www1.ncdc.noaa.gov/pub/data/paleo/paleocean/relative\sea\level/
a Relative Sea Level database

http://www1.ncdc.noaa.gov/pub/data/paleo/paleocean/relative\sea\level/
Can the RSL data be explained GLOBALLY?

Misfit distribution

\[ \mu = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{OBS_i - PRED_i}{\sigma_i} \right)^2 \]

Global RSL data Relative Sea Level - global view

RSL data from file: sealevel.dat

RSL predictions – model ICE5G

Viscosity profile: /10. 0.3 0.3/
LMAX = 128 RES = 48
ITER = 3 Mode = 1
NV = 3 CODE = 0

Number of sites = 392

The data can be explained globally.

Ice model: ICE5G
Repository label: ingv
Viscosity profile: /10. 0.3 0.3/
LMAX = 128 RES = 48 ITER = 3 Mode = 1
NV = 3 CODE = 0
Number of sites = 392
Global RSL data  
Relative Sea Level - local view

Regional and local fit (or misfit)

101 57.0 -77.0 10 RICHMOND GULF QUE. 
Ice model: ICE5G 
Repository label: ing 
Viscosity profile: /10. 0.3 0.3/ 
LMAX =128-RES =48 -ITER =3 –Mode =1 
NV =3 –CODE =0

301 69.0 -35.0 1 KOLA BAY USSR. 
Ice model: ICE5G 
Repository label: ing 
Viscosity profile: /10. 0.3 0.3/ 
LMAX =128-RES =48 -ITER =3 –Mode =1 
NV =3 –CODE =0

501 69.0 -88.3 4 JONATHAN PT. BELIZE 
Ice model: ICE5G 
Repository label: ing 
Viscosity profile: /10. 0.3 0.3/ 
LMAX =128-RES =48 -ITER =3 –Mode =1 
NV =3 –CODE =0
Example: history of RSL across the Hudson Bay
A case study 

the Hudson Bay

Fig. 1. Distribution of six predicted sea level zones resulting from melting of ice sheets 20,000–7,000 yr ago. Within each zone the form of the sea level response is similar. Typical curves indicating the predicted change in sea level relative to the present geoid for each zone are included (RSL means relative sea level). The curves show the wide variety of sea level expressions possible despite the assumption of no change in ocean volume during the past 5,000 yr (no change in ‘eustatic’ sea level) [from Clark and Lingle, 1979].
A case study

the Hudson Bay
A case study: the Hudson Bay

ICE5G thickness until 21 kyrs BP

ICE5G thickness (m)
Equivalent Sea Level

Equivalent Sea Level (ESL) is calculated using the formula:

\[ ESL = \frac{\rho_i V}{\rho_w A_o} \]

Where:
- \( \rho_i \) is the density of the ice
- \( V \) is the volume of the ice
- \( \rho_w \) is the density of seawater
- \( A_o \) is the area of the ocean

Total ESL = 127.40 m
A case study: the Hudson Bay
A case study

the Hudson Bay
Convergence test for the SLE - Ungava Pen.

(viscosity 2:1:1)

RSL data
LMAX=6
12
24
36
72
96

LMAX: Maximum harmonic degree

data (T&P)

A case study                                the Hudson Bay
Convergence test for the SLE - Ungava Pen.

RSL (m)

LMAX = 6

LMAX: Maximum harmonic degree

data (T&P)
A case study: the Hudson Bay

Convergence test for the SLE - Ungava Pen.

LMAX = 6

LMAX: Maximum harmonic degree

RSL (m)

time BP (kyrs)

data (T&P)

(viscosity 2:1:1)
Convergence test for the SLE - Ungava Pen.

LMAX = 6

RSL data
LMAX=6
12
24
36
72
96

(viscosity 2:1:1)

LMAX: Maximum harmonic degree

data (T&P)
A case study: the Hudson Bay

Convergence test for the SLE - Ungava Pen.

RSL (m)

LMAX = 6

(viscosity 2:1:1)

12

24

72: convergence!

RSL data

LMAX=6

12

24

36

72

96

Convergence test for the SLE - Ungava Pen. data (T&P)

LMAX: Maximum harmonic degree

RSL (m)

-120

0

60

120

180

240

300

360

time BP (kyrs)

18

16

14

12

10

8

6

4

2

0
Convergence test for the SLE - Ungava Pen.

LMAX = 6

RSL data
LMAX=6
12
24
36
72
96

86  62.0  -75.0  7 UNGAVA PEN. QUE.

warning... LMAX is strongly load-dependent

data (T&P)

LMAX: Maximum harmonic degree
Convergence test for the SLE - Ungava Pen.

(viscosity 2:1:1, LMAX=72)

RSL data (T&P)

(a)
Convergence test for the SLE - Ungava Pen.

(viscosity 2:1:1, LMAX=72)

RSL (m)

step #0 (eustatic)

data (T&P)

RSL data
SMAX=0
1
2
3
4
5

(a)
A case study

Convergence test for the SLE - Ungava Pen.

(viscosity 2:1:1, LMAX=72)

RSL data
SMAX=0
1
2
3
4
5

data (T&P)

step #0 (eustatic)

Friday, November 11, 2011
Convergence test for the SLE - Ungava Pen.

(viscosity 2:1:1, LMAX=72)

RSL data
S MAX=0
1
2
3
4
5

RSL (m)

step #0 (eustatic)

data (T&P)

(a)
Convergence test for the SLE - Ungava Pen.

(viscosity 2:1:1, LMAX=72)

#5: convergence!

step #0 (eustatic)

data (T&P)

RSL (m)

time BP (kyrs)

(a)
A case study

Convergence test for the SLE - Ungava Pen.

(viscosity 2:1:1, LMAX=72)

number of iterations is load -independent

step #0 (eustatic)

#5: convergence!

data (T&P)

RSL data

SMAX=0

1

2

3

4

5

time BP (kyrs)
Various versions of the SLE - Ungava Pen.

A case study of the Hudson Bay

RSL data
EUS
WOO
ELA
ILN
GSC
Various versions of the SLE - Ungava Pen.

SLE (GSC)

RSL (m)

RSL data
EUS
WOO
ELA
ILN
GSC

Various versions of the SLE - Ungava Pen.

A case study                                the Hudson Bay
A case study

Various versions of the SLE - Ungava Pen.

RSL data
- EUS
- WOO
- ELA
- ILN
- GSC

SLE (GSC)

Eustatic

visco-elasticity

data (T&P)
Various versions of the SLE - Ungava Pen.

SLE (GSC)

Elastic Earth

Eustatic

RSL (m)

time BP (kyrs)

Eustatic data (T&P)

visco-elasticity

RSL data

EUS

WOO

ELA

ILN

GSC

(c)

A case study

the Hudson Bay
Various versions of the SLE - Ungava Pen.

SLE (GSC)

RSL data
EUS
WOO
ELA
ILN
GSC

Elastic Earth

"Woodward"

Eustatic

visco-elasticity

data (T&P)
A case study the Hudson Bay

Various versions of the SLE - Ungava Pen.

Elastic Earth

“Woodward”

ice load neglected

Eustatic

RSL data
EUS
WOO
ELA
ILN
GSC

SLE (GSC)

data (T&P)

visco-elasticity

η μ

Friday, November 11, 2011
A case study: the Hudson Bay

Effect of ice model - Ungava Pen.

ICE-1: Peltier & Andrews, 19XX
ICE-3G: Tushingham & Peltier, 1992

data (T&P)

Time BP (kyrs) vs. Relative Sea Level (m)

ICE-1
ICE-3G
3. Test #0 of the SLE benchmark

a few results

Special topic (2/3)
TEST#0 Earth model description

\[ \mu = \mu(r) \]
\[ \lambda = \lambda(r) \]
\[ \rho = \rho(r) \]
\[ \eta = \eta(r) \]

Lamè constants
Density
Viscosity

✓ (elastic) Lithosphere
✓ Fluid inviscid core
✓ Maxwell Linear viscoelastic rheology
Model M3-L70-V01

(In)compressible, (self)-gravitating

Lithosphere + 3-layer mantle + Fluid Core

Table 1: Model parameters of M3-L70-V01. This is the reference model.

<table>
<thead>
<tr>
<th>Radius (km)</th>
<th>Density (kg/m³)</th>
<th>Rigidity (Pa)</th>
<th>Viscosity (Pa s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6371</td>
<td>3037</td>
<td>$5.0605 \times 10^{10}$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>6301</td>
<td>3438</td>
<td>$7.0363 \times 10^{10}$</td>
<td>$1 \times 10^{21}$</td>
</tr>
<tr>
<td>5951</td>
<td>3871</td>
<td>$1.0549 \times 10^{11}$</td>
<td>$1 \times 10^{21}$</td>
</tr>
<tr>
<td>5701</td>
<td>4978</td>
<td>$2.2834 \times 10^{11}$</td>
<td>$2 \times 10^{21}$</td>
</tr>
<tr>
<td>3480</td>
<td>10750</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
No load is in place. 

**stress-free surface**

A cap load is imposed

**surface vertical stress**

**Goal:** computing displacements, and sea level change...
### List of SLE benchmark contributors / Codes / Methods / not updated / in progress

<table>
<thead>
<tr>
<th>WG4 participant</th>
<th>aka</th>
<th>code name</th>
<th>method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Riccardo Riva</td>
<td>RR</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Valentina Barletta</td>
<td>VB</td>
<td></td>
<td>...</td>
</tr>
<tr>
<td>Volker Klemann</td>
<td>VK</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Zdenek Martinec</td>
<td>ZM</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Paolo Gasperini</strong> (*)?</td>
<td>PG</td>
<td>ABAQUS</td>
<td>FE</td>
</tr>
<tr>
<td>Giorgio Spada</td>
<td>GS</td>
<td>SELEN 3</td>
<td>Pseudo-spectral</td>
</tr>
<tr>
<td>Wouter van der Wal</td>
<td>WW</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Thomas S. James (*)</td>
<td>TJ</td>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>

(* not in WG4)
Sea level change (S)

Colatitude, $\theta$ (deg)

$S = 0$ ka

$S = 1$ ka

$S = 5$ ka

$S = 10$ ka

$S$ (m)

$GS$ (CM frame)

$VB$

$VK$

$WW$

$TJ&KS$

$\theta$ (deg)

$S$ (m)

$0$  $10$  $20$  $50$  $100$

$0$  $10$  $20$  $50$  $100$

$0$  $10$  $20$  $50$  $100$

$0$  $10$  $20$  $50$  $100$
Vertical displacement (U)

Colatitude, \( \theta \) (deg)

GS (CM reference frame)
- VB
- VK
- WW
- TJ&KS

Colatitude, \( \theta \) (deg)

Colatitude, \( \theta \) (deg)

Colatitude, \( \theta \) (deg)

Colatitude, \( \theta \) (deg)
Sea surface variation (N)

Colatitude, $\theta$ (deg)

$t=0$ ka

$t=1$ ka

$t=5$ ka

$t=10$ ka

Colatitude, $\theta$ (deg)
Lack of symmetry imposes numerical integration in latitude: 
tricky integrals of (oscillating) Associated Legendre functions
Special topic (3/3):

present-day geodetic variations and the SLE
Is glacio-isostatic adjustment (GIA) affecting the Earth shape today?

Yes, in many respects:

✓ present-day sea level variations on a global scale and regional,
✓ vertical movements and geoid height variations,
✓ “local” sea level change at tide-gauges,
✓ rate of change of harmonic components of geo-potential,
✓ ground deformation (vertical + horizontal movements),
✓ stress field variations in deglaciated areas...
✓ ...

All these effects are referred to as “Geodetic variations”; they can be studied by the same fundamental equation used so far (the SLE).

We only consider some of these issues...
Solving the SLE we obtain the rate of present-day sea level change:

\[ \dot{S}(\omega, t_p) = \frac{dS(\omega, t)}{dt} \bigg|_{t=t_p} \]

but also the rate of vertical displacement ("vertical velocity")

\[ \dot{U}(\omega, t_p) = \frac{dU(\omega, t)}{dt} \bigg|_{t=t_p} \]

and the rate of sea surface height change:

\[ \dot{N}(\omega, t_p) = \dot{U}(\omega, t_p) + \dot{S}(\omega, t_p) \]

These rates amount to several mm/yr, a large signal geodetically.
Role of GIA today

Sea level change

Rate of sea level change today

- Ice model: ICE5G
- Viscosity profile: /10. 0.4 0.4/

- \( \dot{S} \) (mm/yr)

- LMAX=128 - RES=48 - NV=3 - CODE=2 - MODE=1 - ITER=5
Role of GIA today

Geoid height variation

Rate of geoid height variation today

- Ice model: ICE5G
- Viscosity profile: /10. 0.4 0.4/

- LMAX=128 - RES=48 - NV=3 - CODE=2 - MODE=1 - ITER=5
Role of GIA today

Vertical velocity

Rate of vertical displacement today
- Ice model: ICE5G
- Viscosity profile: /10. 0.4 0.4/

![Map showing rate of vertical displacement](image)

- LMAX=128
- RES=48
- NV=3
- CODE=2
- MODE=1
- ITER=5

Friday, November 11, 2011
1. General ideas about GIA

Example #3: geoid subsidence in the MED

Rate of sea surface height variation today (MED)
GIA correction to altimetry data

Nine “perturbed” viscosity profiles assuming ICE-5G (Peltier, 2004)

preferred rate: -0.30 +/- 0.10 mm/yr
Role of GIA today

Rate of sea level change today – MEDITERRANEAN

- Ice model: ICE5G
- Viscosity profile: /10. 0.3 0.3/

- LMAX=128
- RES=48
- NV=3
- CODE=0
- MODE=1
- ITER=3

mm/yr
Rate of vertical uplift today – MEDITERRANEAN

Ice model: ICE5G
Viscosity profile: /10. 0.3 0.3/

Role of GIA today

Mediterranean Sea

- LMAX=128 - RES=48 - NV=3 - CODE=0 - MODE=1 - ITER=3
Permanent Service for Mean Sea Level

PSML 75th Anniversary

PSML organised three events in celebration of its 75th Anniversary. The highlight of the year was ‘Liverpool, Home of Sea Level Science: Sea Level Rise and Climate Change’, a session at the September 2008 meeting of the British Association Festival of Science in Liverpool. A number of distinguished speakers from the UK and abroad presented an overview of the measurements and causes of sea level change, as well as the history of this research at Liverpool.

In addition, PSML gathered scientific experts to discuss ongoing research into sea level. It planned a special Interdivision Session at the 2008 Assembly of the European Geophysical Union (abstract and contributed presentations.) Furthermore, it sponsored the Geological Society’s William Smith Meeting 2008. The meeting was titled “Observations and Causes of Sea-Level Changes on Millennial to Decadal Timescales” and occurred 1-2 September 2008.

PSML commentaries

We have started a page in which ‘commentaries’ of records in the data set can be posted for the interest of other users of the data set - more information.

Recent Events demonstrate need for sea level data

Two recent events highlight the need for sharing information on sea-level changes internationally: the Sumatra tsunami of December 2004 and the hurricane Katrina floods of August 2005. There are also scientific and public concerns about climate change and sea-level rise. The PSML is the global data bank for long-term sea-level information from tide gauges. It operates with the support of the International Council for Science. PSML's data holdings now total more than 54,000 station-years from around 2,000 stations, an increase of over 1,600 station-years during 2005-06.

International Polar Year

The International Polar Year (IPY, 2007-08) is focusing attention on making and using more

http://www.pol.ac.uk
is sea level “rising” or “falling”?

```
"psmslp_new.dat" using 3
```
### Role of GIA today

2008.05.08 time=08.45.50  
Ice model: ice5g.dat  
Number of mantle layers: 3  
Model code (see TABOO User guide): 0  
Viscosity model: visco.dat  
Thickness of the lithosphere (km): '65.'  
Viscosity (bottom-to-top, Haskell units) '10.'  
Viscosity (bottom-to-top, Haskell units) '0.3'  
Viscosity (bottom-to-top, Haskell units) '0.3'  
SLE iterations: 3  
SLE mode of solution: 1  
Maximum harmonic degree: 128  
Tegmark resolution: 48  

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<th>error</th>
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<td>-0.830</td>
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The rate of GLOBAL SEA LEVEL RISE is estimated through the analysis of "LONG" series of TG data!

Current estimates are collected in this table.

Table 3.1
Recent Determinations of Global Sea Level Rise from Tide Gauge Data

<table>
<thead>
<tr>
<th>Author</th>
<th>Estimate (mm/yr)</th>
<th>Comments</th>
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<tbody>
<tr>
<td>Peltier and Tushingham (1989, 1991)</td>
<td>$2.4 \pm 0.9^a$</td>
<td>Global data</td>
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<td>Barnett (1990)</td>
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<td>Global data</td>
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<td>Nakiboglu and Lambeck (1990)</td>
<td>$1.15 \pm 0.38$</td>
<td>Global data</td>
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<td>Trupin and Wahr (1990)</td>
<td>$1.75 \pm 0.13$</td>
<td>Global data</td>
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<td>Douglas (1991)</td>
<td>$1.8 \pm 0.1$</td>
<td>Global data</td>
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<tr>
<td>Shennan and Woodworth (1992)</td>
<td>$1.0 \pm 0.15$</td>
<td>U.K. and Europe</td>
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<td>Mitrovica and Davis (1995)</td>
<td>1.1–1.6</td>
<td>Global data</td>
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<td>$1.5 \pm 0.3$</td>
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<td>Peltier (1996)</td>
<td>$1.94 \pm 0.6^a$</td>
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<td>Douglas (1997)</td>
<td>$1.8 \pm 0.1$</td>
<td>Global data</td>
</tr>
</tbody>
</table>

$^a$ Standard deviation of trends about their mean. The formal SE is a few tenths.
2: Stokes coefficients variations associated with GIA

By simple arguments (unpublished manuscript), we have obtained the Stokes coefficients variations (GRACE conventions) with:

\[
\begin{align*}
\bar{c}_{lm}^G &= +(-1)^m \sqrt{2 - \delta_{0m}} \frac{Re(N_{lm})}{a} \\
\bar{s}_{lm}^G &= -(-1)^m \sqrt{2 - \delta_{0m}} \frac{Im(N_{lm})}{a},
\end{align*}
\]

with:

\(N_{lm}\) Harmonics of the sea surface elevation change,
\(a\) Reference radius of the Earth.
Rate of change of the fully–normalized Stokes coefficients

Ice model: ICE5G –Viscosity profile: /4.0 0.4 0.4/ –LMAX=128 –RES=44 –NV=3 –CODE=2 –MODE=1 –ITER=3
6.4. GIA predictions with SELEN 2.9

Figure 6.5: Present-day rate of change of the fully normalized Stokes coefficients. (a) Rate of change of the C_{lm} coefficients, (b) Rate of change of the S_{lm} coefficients. Red: ICE-5G(VM2), Green: ICE-3G(TP), Black: ANU05(KL).

from the Ph.D. thesis of

Sofie Louise Sandberg Sorensen

“Changes of the Greenland ice sheet - derived from ICESat and GRACE data”

Figure 6.5: Present-day rate of change of the fully normalized Stokes coefficients.

(a) Rate of change of the C_{lm} coefficients,

(b) Rate of change of the S_{lm} coefficients.

Red: ICE-5G(VM2),

Green: ICE-3G(TP),

Black: ANU05(KL).