

Chapter 1

Reminder - balance laws in the reference configuration

For continuum mechanics and thermodynamics of solids, one often benefits from the fact that a physically relevant and well defined reference configuration is meaningful (e.g. the initial stress-free state or a state with a known geometry and pre-stress). Therefore the referential (Lagrangian) formulation of the balance and conservation equations plays an important role. Let us re-derive the basic balance laws from their integral formulation in the actual configuration, as it is usually done.

Balance of mass

Considering a body in actual configuration (denoted as $\mathcal{B}(t) \subset \mathbb{R}^3$) and an arbitrary (sufficiently smooth) material volume $\omega(t)$ in the actual configuration, its subset. The balance of mass for a single-component continuum can be postulated as follows

$$\frac{d}{dt} \int_{\omega(t)} \rho(\mathbf{x}, t) dx = 0, \quad \forall \omega(t) \subset \mathcal{B}(t), \omega \text{ material} \quad (1.0.1)$$

where $\rho(\mathbf{x}, t)$ denotes the density in the actual configuration. Applying the pull-back to reference configuration by the mapping $\mathbf{x} = \chi(\mathbf{X}, t)$, we get

$$\frac{d}{dt} \int_{\Omega} \underbrace{\rho(\chi(\mathbf{X}, t), t)}_{\stackrel{\text{def}}{=} \rho(\mathbf{X}, t)} \underbrace{\det \mathbb{F}(\mathbf{X}, t)}_{=dx} dX = 0. \quad (1.0.2)$$

Now since the pull-back of the volume $\omega(t)$ is time independent material volume Ω in the reference configuration ($\omega(t) = \chi(\Omega, t)$ i.e. $\Omega = \chi^{-1}(\omega(t), t)$), the time derivative commutes with the integral and we obtain

$$0 = \int_{\Omega} \frac{\partial}{\partial t} (\rho(\mathbf{X}, t) \det \mathbb{F}(\mathbf{X}, t)) dX. \quad (1.0.3)$$

By localization argument, we argue that

$$\rho(\mathbf{X}, t) \det \mathbb{F}(\mathbf{X}, t) = f(\mathbf{X}), \quad (1.0.4)$$

and taking the limit $t \rightarrow t_0$, we get $f(\mathbf{X}) = \rho_R(\mathbf{X})$, where $\rho_R(\mathbf{X})$ denotes the density field at $t = 0$. So the local form of mass balance in the reference configuration takes the following form

$$\boxed{\rho(\mathbf{X}, t) \det \mathbb{F}(\mathbf{X}, t) = \rho_R(\mathbf{X})}, \quad (1.0.5)$$

for all \mathbf{X} in the reference configuration and all t .

Balance of linear momentum

In the actual configuration, the integral form of the balance of linear momentum for a non-polar single-component material is postulated as follows

$$\frac{d}{dt} \int_{\omega(t)} \rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) dx = \int_{\partial\omega(t)} \mathbb{T}(\mathbf{x}, t) \mathbf{n}(\mathbf{x}, t) ds + \int_{\omega(t)} \rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) dx, \quad \forall \omega(t) \subset \mathcal{B}(t), \omega \text{ material} \quad (1.0.6)$$

where \mathbf{v} is velocity, \mathbb{T} denotes the Cauchy stress tensor and \mathbf{b} is the body force (e.g. gravity). Using again the pullback to the reference configuration and the transformation of surface element

$$\mathbf{n} ds \stackrel{\text{def}}{=} d\mathbf{s} = \text{cof} \mathbb{F} d\mathbf{S} = \det \mathbb{F} \mathbb{F}^{-T} \mathbf{N} dS, \quad (1.0.7)$$

yields

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \underbrace{\rho(\chi(\mathbf{X}, t), t) \det \mathbb{F}(\mathbf{X}, t)}_{\rho_R(\mathbf{X})} \underbrace{\mathbf{v}(\chi(\mathbf{X}, t), t)}_{\stackrel{\text{def}}{=} \mathbf{V}(\mathbf{X}, t)} dX &= \int_{\partial\Omega} \mathbb{T}(\chi(\mathbf{X}, t), t) \det \mathbb{F}(\mathbf{X}, t) \mathbb{F}^{-T}(\mathbf{X}, t) \mathbf{N} dS \\ &+ \int_{\Omega} \underbrace{\rho(\chi(\mathbf{X}, t), t) \det \mathbb{F}(\mathbf{X}, t)}_{\rho_R(\mathbf{X})} \underbrace{\mathbf{b}(\chi(\mathbf{X}, t))}_{\stackrel{\text{def}}{=} \mathbf{B}(\mathbf{X}, t)} dX \end{aligned} \quad (1.0.8)$$

Defining the first Piola-Kirchhoff stress $\mathbb{T}^{(1)}$ by

$$\mathbb{T}^{(1)}(\mathbf{X}, t) \stackrel{\text{def}}{=} \det \mathbb{F}(\mathbf{X}, t) \mathbb{T}(\chi(\mathbf{X}, t), t) \mathbb{F}^{-T}(\mathbf{X}, t), \quad (1.0.9)$$

the pull-back of the integral momentum balance thus reads

$$\frac{d}{dt} \int_{\Omega} \rho_R(\mathbf{X}) \mathbf{V}(\mathbf{X}, t) dX = \int_{\partial\Omega} \mathbb{T}^{(1)}(\mathbf{X}, t) \mathbf{N} dS + \int_{\Omega} \rho_R(\mathbf{X}) \mathbf{B}(\mathbf{X}, t). \quad (1.0.10)$$

Switching the order of spatial integration and time differentiation in the first term, by applying Gauss' theorem, in the second, and by a localitization argument, we obtain the following referential local form of the linear momentum balance

$$\rho_R(\mathbf{X}) \frac{\partial \mathbf{V}(\mathbf{X}, t)}{\partial t} = \text{Div} \mathbb{T}^{(1)}(\mathbf{X}, t) + \rho_R(\mathbf{X}) \mathbf{B}(\mathbf{X}, t), \quad (1.0.11)$$

where Div denotes the divergence operator in the reference frame, i.e. for a second order tensor $(\text{Div} \mathbb{T}^{(1)})^i = \frac{\partial \mathbb{T}^{(1)ij}}{\partial X^j}$.

Balance of angular momentum

Since we consider a non-polar continuum, we know that the momentum balance in the actual configuration implies the symmetry of the Cauchy stress, i.e. relation

$$\mathbb{T}(\mathbf{x}, t) = \mathbb{T}^T(\mathbf{x}, t). \quad (1.0.12)$$

Using the definition of the first Piola Kirchhoff stress

$$\mathbb{T}^{(1)}(\mathbf{X}, t) = \det \mathbb{F}(\mathbf{X}, t) \mathbb{T}(\chi(\mathbf{X}, t), t) \mathbb{F}^{-T}(\mathbf{X}, t), \quad (1.0.13)$$

implies that $\mathbb{T}^{(1)}$ is not symmetric in general, but instead, the following holds

$$\mathbb{T}^{(1)}(\mathbf{X}, t) \mathbb{F}^T(\mathbf{X}, t) = \mathbb{F}(\mathbf{X}, t) (\mathbb{T}^{(1)})^T(\mathbf{X}, t) \quad (1.0.14)$$

Balance of energy

The integral form of the balance of energy for a single-component non-polar material is postulated in the following form

$$\begin{aligned} \frac{d}{dt} \int_{\omega(t)} \rho(\mathbf{x}, t) \left(e(\mathbf{x}, t) + \frac{1}{2} |\mathbf{v}(\mathbf{x}, t)|^2 \right) dx &= \int_{\partial\omega(t)} \mathbf{v}(\mathbf{x}, t) \cdot \mathbb{T}(\mathbf{x}, t) \mathbf{n} ds - \int_{\partial\omega(t)} \mathbf{q}(\mathbf{x}, t) \cdot \mathbf{n} ds + \int_{\omega(t)} \rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) dx \\ &+ \int_{\omega(t)} \rho(\mathbf{x}, t) r(\mathbf{x}, t) dx, \quad \forall \omega(t) \subset \mathcal{B}(t), \omega \text{ material} \end{aligned} \quad (1.0.15)$$

where e denotes the specific internal energy and r represents the external energy sources (e.g. radiative heating). The pull-back to the reference configuration now yields

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \underbrace{\rho(\chi(\mathbf{X}, t), t) \det \mathbb{F}(\mathbf{X}, t)}_{\rho_R(\mathbf{X})} \left(\underbrace{e(\chi(\mathbf{X}, t), t)}_{\stackrel{\text{def}}{=} e_R(\mathbf{X}, t)} + \frac{1}{2} \underbrace{|\mathbf{v}(\chi(\mathbf{X}, t), t)|^2}_{= \mathbf{V}(\mathbf{X}, t)} \right) dX &= \int_{\partial\Omega} \underbrace{\mathbf{v}(\chi(\mathbf{X}, t), t)}_{\mathbf{V}(\mathbf{X}, t)} \cdot \underbrace{\mathbb{T}(\chi(\mathbf{X}, t), t) \det \mathbb{F}(\mathbf{X}, t) \mathbb{F}^{-T}(\mathbf{X}, t) \mathbf{N}}_{\mathbb{T}^{(1)}(\mathbf{X}, t)} dS \\ - \int_{\partial\Omega} \mathbf{q}(\chi(\mathbf{X}, t), t) \cdot \det \mathbb{F}(\mathbf{X}, t) \mathbb{F}^{-T}(\mathbf{X}, t) \mathbf{N} dS &+ \int_{\Omega} \underbrace{\rho(\chi(\mathbf{X}, t), t) \det \mathbb{F}(\mathbf{X}, t)}_{\rho_R(\mathbf{X})} \underbrace{\mathbf{b}(\chi(\mathbf{X}, t), t) \cdot \mathbf{v}(\chi(\mathbf{X}, t), t)}_{\mathbf{B}(\mathbf{X}, t) \cdot \mathbf{V}(\mathbf{X}, t)} dX \\ + \int_{\Omega} \underbrace{\rho(\chi(\mathbf{X}, t), t) \det \mathbb{F}(\mathbf{X}, t)}_{\rho_R(\mathbf{X})} \underbrace{r(\chi(\mathbf{X}, t), t)}_{\stackrel{\text{def}}{=} R(\mathbf{X}, t)} dX & \end{aligned}$$

We use the identity

$$\mathbf{q}(\chi(\mathbf{X}, t), t) \cdot \det \mathbb{F}(\mathbf{X}, t) \mathbb{F}^{-T}(\mathbf{X}, t) \mathbf{N} = \det \mathbb{F}(\mathbf{X}, t) \mathbb{F}^{-1}(\mathbf{X}, t) \mathbf{q}(\chi(\mathbf{X}, t), t) \cdot \mathbf{N}, \quad (1.0.16)$$

and defining the referential energy flux \mathbf{Q} by

$$\boxed{\mathbf{Q}(\mathbf{X}, t) \stackrel{\text{def}}{=} \det \mathbb{F}(\mathbf{X}, t) \mathbb{F}^{-1}(\mathbf{X}, t) \mathbf{q}(\chi(\mathbf{X}, t), t)}, \quad (1.0.17)$$

we rewrite the energy balance as

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \rho_R(\mathbf{X}) \left(e_R(\mathbf{X}, t) + \frac{1}{2} |\mathbf{V}(\mathbf{X}, t)|^2 \right) dX &= \int_{\partial\Omega} \mathbf{V}(\mathbf{X}, t) \cdot \mathbb{T}^{(1)}(\mathbf{X}, t) \mathbf{N} dS - \int_{\partial\Omega} \mathbf{Q}(\mathbf{X}, t) \cdot \mathbf{N} dS \\ + \int_{\Omega} \rho_R(\mathbf{X}) \mathbf{B}(\mathbf{X}, t) \cdot \mathbf{V}(\mathbf{X}, t) dX &+ \int_{\Omega} \rho_R(\mathbf{X}) R(\mathbf{X}, t) dX. \end{aligned} \quad (1.0.18)$$

Switching the integration and differentiation in the first term, applying Gauss' theorem in the second and third, any by a localization argument, we arrive at the following local form of the energy balance in the reference configuration

$$\boxed{\rho_R(\mathbf{X}) \frac{\partial}{\partial t} \left(e_R(\mathbf{X}, t) + \frac{1}{2} |\mathbf{V}(\mathbf{X}, t)|^2 \right) = \text{Div} \left(\mathbf{V}(\mathbf{X}, t) \cdot \mathbb{T}^{(1)}(\mathbf{X}, t) \right) - \text{Div} \mathbf{Q}(\mathbf{X}, t) + \rho_R(\mathbf{X}) \mathbf{B}(\mathbf{X}, t) \cdot \mathbf{V}(\mathbf{X}, t) + \rho_R(\mathbf{X}, t) R(\mathbf{X}, t)}. \quad (1.0.19)$$

Naturally, we are also interested in the internal energy balance, which we obtain in the standard manner by subtracting the kinetic energy balance. First, we note that

$$\left(\text{Div} \left(\mathbf{V} \cdot \mathbb{T}^{(1)} \right) \right)^i = \frac{\partial}{\partial \mathbf{X}^J} \left(V_i \mathbb{T}^{(1) iJ} \right) = \frac{\partial V_i}{\partial \mathbf{X}^J} (\mathbb{T}^{(1)})^{iJ} + V_i \frac{\partial \mathbb{T}^{(1) iJ}}{\partial \mathbf{X}^J}$$

Since

$$\frac{\partial V_i}{\partial \mathbf{X}^J} = \frac{\partial}{\partial \mathbf{X}^J} \left(\frac{\partial \chi_i}{\partial t} \right) = \frac{\partial}{\partial t} \left(\frac{\partial \chi_i}{\partial \mathbf{X}^J} \right) = \frac{\partial}{\partial t} \mathbb{F}_{iJ} = \dot{\mathbb{F}}_{iJ}, \quad (1.0.20)$$

we get

$$\left(\text{Div}(\mathbf{V} \cdot \mathbb{T}^{(1)})\right)^i = \dot{\mathbb{F}}_{iJ}(\mathbb{T}^{(1)})^{iJ} + \mathbf{V}_i \frac{\partial \mathbb{T}^{(1)iJ}}{\partial \mathbf{X}^J} = \dot{\mathbb{F}} : \mathbb{T}^{(1)} + \mathbf{V} \cdot \text{Div} \mathbb{T}^{(1)}. \quad (1.0.21)$$

Taking a scalar product of the momentum balance (1.0.11) with velocity \mathbf{V} yields

$$\rho_R \frac{\partial}{\partial t} \left(\frac{|\mathbf{V}|^2}{2} \right) = \mathbf{V} \cdot \text{Div} \mathbb{T}^{(1)} + \rho_R \mathbf{B} \cdot \mathbf{V} \quad (1.0.22)$$

Subtracting this kinetic energy balance from the total energy balance (1.0.19), using also (1.0.21) yields the desired internal energy balance in the form

$$\boxed{\rho_R(\mathbf{X}) \frac{\partial e_R(\mathbf{X}, t)}{\partial t} = \dot{\mathbb{F}} : \mathbb{T}^{(1)} - \text{Div} \mathbf{Q}(\mathbf{X}, t) + \rho_R(\mathbf{X}) R(\mathbf{X}, t)} \quad (1.0.23)$$

Balance of entropy

The integral form of entropy balance for a single-component continuum in the actual configuration is postulated in the following form

$$\frac{d}{dt} \int_{\omega(t)} \rho(\mathbf{x}, t) \eta(\mathbf{x}, t) dx = - \int_{\partial\omega(t)} \mathbf{j}_\eta(\mathbf{x}, t) \cdot \mathbf{n} ds + \int_{\omega(t)} \rho(\mathbf{x}, t) r_\eta(\mathbf{x}, t) dx + \int_{\omega(t)} \xi(\mathbf{x}, t) dx, \quad \forall \omega(t) \subset \mathcal{B}(t), \omega \text{ material},$$

where η is the specific entropy, \mathbf{j}_η is the entropy flux, r_η is the outer entropy supply, and ξ is the entropy production, which must be non-negative in order to comply with the second law of thermodynamics. Pull-back to the reference configuration reads

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \underbrace{\rho(\chi(\mathbf{X}, t), t)}_{\rho_R(\mathbf{X})} \underbrace{\det \mathbb{F}(\mathbf{X}, t) \eta(\chi(\mathbf{X}, t), t)}_{\stackrel{\text{def}}{=} \eta_R(\mathbf{X}, t)} dX &= - \int_{\partial\Omega} \mathbf{j}_\eta(\chi(\mathbf{X}, t), t) \cdot \det \mathbb{F}(\mathbf{X}, t) \mathbb{F}^{-T}(\mathbf{X}, t) \mathbf{N} dS \\ &+ \int_{\Omega} \rho(\chi(\mathbf{X}, t), t) r_\eta(\chi(\mathbf{X}, t), t) dX + \int_{\Omega} \det \mathbb{F}(\mathbf{X}, t) \xi(\chi(\mathbf{X}, t), t) dX, . \end{aligned}$$

Analogously to the energy balance, we use the identity

$$\mathbf{j}_\eta(\chi(\mathbf{X}, t), t) \cdot \det \mathbb{F}(\mathbf{X}, t) \mathbb{F}^{-T}(\mathbf{X}, t) \mathbf{N} = \det \mathbb{F}(\mathbf{X}, t) \mathbb{F}^{-1}(\mathbf{X}, t) \mathbf{j}_\eta(\chi(\mathbf{X}, t), t) \cdot \mathbf{N}, \quad (1.0.24)$$

and defining the referential entropy flux \mathbf{J}_η by

$$\boxed{\mathbf{J}_\eta(\mathbf{X}, t) \stackrel{\text{def}}{=} \det \mathbb{F}(\mathbf{X}, t) \mathbb{F}^{-1}(\mathbf{X}, t) \mathbf{j}_\eta(\chi(\mathbf{X}, t), t)}, \quad (1.0.25)$$

using Gauss' theorem on the surface integral, and using localization argument, we obtain the following local form of the entropy balance

$$\boxed{\rho_R(\mathbf{X}) \frac{\partial \eta_R(\mathbf{X}, t)}{\partial t} = - \text{Div} \mathbf{J}_\eta(\mathbf{X}, t) + \rho_R(\mathbf{X}) R_\eta(\mathbf{X}, t) + \det \mathbb{F}(\mathbf{X}, t) \xi_R(\mathbf{X}, t)}. \quad (1.0.26)$$

The entropy production must be non-negative in order to comply with the second law of thermodynamics, to the above balance is supplemented with the assumption

$$\boxed{\xi_R \geq 0}. \quad (1.0.27)$$