Homework nr. 1 - deadline 26.3.

1. Compute directional derivatives of the principal invariants of matrix $\mathbb{A}$ in the direction $\mathbb{B}$, i.e. for

$$I_1(\mathbb{A}) \stackrel{\text{def}}{=} \text{Tr}\mathbb{A} \;, \tag{1a}$$

$$I_2(\mathbb{A}) \stackrel{\text{def}}{=} \frac{1}{2}((\text{Tr}\mathbb{A})^2 - \text{Tr}(\mathbb{A}^2)) \;, \tag{1b}$$

$$I_3(\mathbb{A}) \stackrel{\text{def}}{=} \det\mathbb{A} \;, \tag{1c}$$

compute $\frac{\partial I_\alpha}{\partial \mathbb{A}}[\mathbb{B}]$, using the tensor analysis calculus you know from the continuum mechanics course. Verify that you obtain the same also component-wise by first computing

$$\frac{\partial I_\alpha(\mathbb{A})}{\partial \mathbb{A}_{ij}} \qquad \alpha, i, j \in \{1, 2, 3\},$$

and then constructing a linear form w.r.t. $\mathbb{B}$ by contracting, i.e. evaluating

$$\sum_{i,j=1}^{3} \frac{\partial I_\alpha(\mathbb{A})}{\partial \mathbb{A}_{ij}} \mathbb{B}_{ij}$$

.

2. Consider the following transformation, denoted as change of observer, which consists of time-dependent translation of origins $\mathbf{o}$ and $\mathbf{o}^*$ and mutual rotation of two Cartesian coordinate frames $\{\mathbf{e}_i\}_{i=1}^3$ and $\{\mathbf{e}_i^*\}_{i=1}^3$.

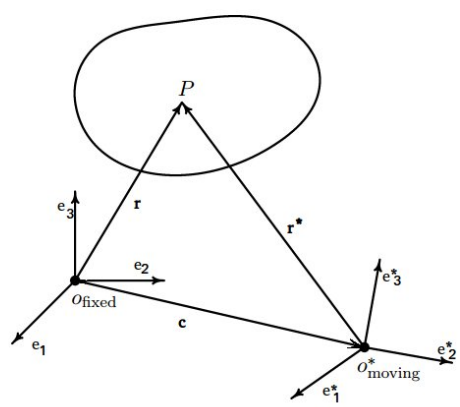

Figure 1:

Let us assume the first observer is fixed, while the other is time-dependent and given by

$$\mathbf{o}^*(t) = \mathbf{o} + \mathbf{c}(t) \;, \tag{2a}$$

$$\mathbf{e}_i^*(t) = \mathbf{Q}_{ij}(t)\mathbf{e}_j \;. \tag{2b}$$

where $\mathbf{Q}(t)\mathbf{Q}^T(t) = \mathbf{Q}^T(t)\mathbf{Q}(t) = \mathbf{I}$, $\det\mathbf{Q}(\mathbf{t}) = \mathbf{1}$, $\forall t$.

Then any vector $\mathbf{a}$ or any tensor $\mathbb{A}$ we can be represented in terms of coordinates with respect to both frames and it must hold

$$\mathbf{a} = \mathrm{a}_i\mathbf{e}_i = \mathrm{a}_i^*\mathbf{e}_i^* \ , \tag{3a}$$
$$\mathbb{A} = \mathbb{A}_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \mathbb{A}_{ij}^*\mathbf{e}_i^* \otimes \mathbf{e}_j^* \ , \tag{3b}$$

which implies that the coordinates of vectors and tensors during the change of observer transform as

$$\mathrm{a}_i^* = \mathbf{Q}_{ij}\mathrm{a}_j \ , \tag{4a}$$
$$\mathbb{A}_{ij}^* = \mathbf{Q}_{ik}\mathbf{Q}_{jl}\mathbb{A}_{kl} \ . \tag{4b}$$

The position vectors $\mathbf{r}, \mathbf{r}^*$ to a point P are not the same geometric objects, instead they satisfy (see the figure)

$$\mathbf{r} = \mathbf{r}^* + \mathbf{c} \ , \tag{5}$$

or in terms of coordinates

$$x_i\mathbf{e}_i = x_j^*\mathbf{e}_j^* + \mathrm{c}_i\mathbf{e}_i \ . \tag{6}$$

Taking a scalar product of the above relation with $\mathbf{e}_k^*$ yields (using the orthonormality both bases and relation (2b)) the relation among the spatial coordinates of point P w.r.t. the two frames:

$$x_k^* = \mathbf{Q}_{kj}(t)(x_j - c_j(t)) \ , \tag{7}$$

where we explicitly highlighted the time-dependence arising from the time-dependence of the second observer frame.

TASK: Examine transformation properties under the change of observer of Cartesian components of kinematic tensors of continuum mechanics: the deformation gradient $\mathbb{F}$, Right Cauchy-Green tensor $\mathbb{C} = \mathbb{F}^T\mathbb{F}$, Left Cauchy-Green tensor $\mathbb{B} = \mathbb{F}\mathbb{F}^T$, velocity gradient $\mathbb{L} = \mathrm{grad}\mathbf{v}$, symmetric part of the velocity gradient $\mathbb{D} = \mathrm{sym}(\mathbb{L})$, antisymmetric part of the velocity gradient $\mathbb{W} = \mathrm{skew}(\mathbb{L})$, Green-Saint Venant tensor $\mathbb{E} = \frac{1}{2}(\mathbb{C} - \mathbb{I})$, Euler-Almansi tensor $= \frac{1}{2}(\mathbb{I} - \mathbb{B}^{-1})$ and the relative deformation gradient $\mathbb{F}_t(\mathbf{x}, \tau) = \mathbb{F}(\mathbf{X}, \tau)\mathbb{F}^{-\mathbf{1}}(\mathbf{X}, \mathbf{t})|_{\mathbf{X}=\boldsymbol{\chi}^{-\mathbf{1}}(\mathbf{x},\mathbf{t})}$.

HINT: Use the definition of the components of $\mathbb{F}$, i.e. $\mathbb{F}_{iK} = \frac{\partial \boldsymbol{\chi}^i}{\partial \mathbf{X}^K}$ and express relationship between χ^* and χ with respect to **one basis**, e.g. $\{\mathbf{e_i}\}_{\mathbf{i=1}}^{\mathbf{3}}$

$$\boldsymbol{\chi}_i^*(\mathbf{X}, t) = \mathbf{Q}_{ij}(t)\boldsymbol{\chi}_j(\mathbf{X}, t) + c_i(t) \, . \tag{8}$$

The transformation properties of components of $\mathbb{C}$, $\mathbb{B}$, $\mathbb{E}$, follow from the definitions. Concerning $\mathbb{L}$, $\mathbb{D}$, $\mathbb{W}$, start from the definition of material velocity using the transformation rule (7).