Chapter 2

Objectivity, material frame indifference and material symmetry

Plan: We will introduce two important classes of transformations - *change of observer* and *change of reference configuration*, we will recall effects of these two types of transformations on the kinematic quantities of continuum mechanics, and finally, we will use these two classes to define a notion of *material frame indifference* and of *material symmetry* and the associated symmetry group for constitutive functions. These two notions will then allow us to employ representation theorems and significantly reduce the possible class of free energy functions for isotroic hyperelastic materials.

Change of observer

Let us define a reference (fixed in time) *observer* $\mathscr{O}$ as the set

$$\mathscr{O} \stackrel{\text{def}}{=} \{\mathbf{o}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}, \tag{2.0.1}$$

i.e. origin $\mathbf{o}$ and global orthogonal coordinate system defined by unit vectors $\boldsymbol{e}_i$, $i = 1,2,3$. Let us note that is probably the simplest, but not the only possible way how to do that. Let us now consider another observer $\mathscr{O}^*$, moving with respect to observer $\mathscr{O}$

$$\mathscr{O}^* \stackrel{\text{def}}{=} \{\mathbf{o}^*, \boldsymbol{e}_1^*, \boldsymbol{e}_2^*, \boldsymbol{e}_3^*\}, \tag{2.0.2}$$

where the two observers are related via translation vector $\mathbf{c}(t)$ and rotation matrix $\mathbf{Q}(t)$

$$\mathbf{o}^*(t) = \mathbf{o} + \mathbf{c}(t), \quad \text{and} \quad \boldsymbol{e}_i^*(t) = \mathbf{Q}_i^{\ j}(t)\boldsymbol{e}_j, \qquad \text{where} \quad \mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbb{I}. \tag{2.0.3}$$

The latter property of $\mathbf{Q}$ follows from assumed ortho-normality of both coordinate bases:

$$\delta_{ij} = (\boldsymbol{e}_i^*, \boldsymbol{e}_j^*) = (\mathbf{Q}_i^{\ k}\boldsymbol{e}_k, \mathbf{Q}_j^{\ l}\boldsymbol{e}_l) = \mathbf{Q}_i^{\ k}\mathbf{Q}_j^{\ l}\underbrace{(\boldsymbol{e}_k, \boldsymbol{e}_l)}_{=\delta_{kl}} = \mathbf{Q}_{il}\mathbf{Q}_j^{\ l} = \mathbf{Q}_{il}(\mathbf{Q}^T)^l_{\ j}. \tag{2.0.4}$$

As a homework, you showed that under the change of observer the deformation gradient $\mathbb{F}$ transforms (when considered as a matrix) as follows

$$\boxed{\mathbb{F}^* = \mathbf{Q}\mathbb{F}} \qquad \text{in the sense} \quad \mathbb{F}^{*i}_{\ \ K} = \mathbf{Q}^i_{\ j}\mathbb{F}^j_{\ K}. \tag{2.0.5}$$

Note: this transformation rule followed either from the definition of $\mathbb{F}$, which for the starred observer gives

$$(\mathbb{F}^*)^i_{\ K} = \frac{\partial \boldsymbol{\chi}^{*i}}{\partial \mathbf{X}^K} = \frac{\partial\left(\mathbf{Q}^i_{\ j}\boldsymbol{\chi}^j + \mathbf{c}^i(t)\right)}{\partial \mathbf{X}^J} = \mathbf{Q}^i_{\ j}\frac{\partial \boldsymbol{\chi}^j}{\partial \mathbf{X}^K} = \mathbf{Q}^i_{\ j}\mathbb{F}^j_{\ K}. \tag{2.0.6}$$

As a consequence, you arrived at the following transformation rules for other important kinematic quantities

left Cauchy Green tensor	$\boxed{\mathbb{C}^* = \mathbb{C}}$	in the sense	$\mathbb{C}^*_{IJ} = \mathbb{C}_{IJ}$
right Cauchy Green tensor	$\boxed{\mathbb{B}^* = \mathbf{Q}\mathbb{B}\mathbf{Q}^T}$	in the sense	$\mathbb{B}^{ij} = \mathbf{Q}^i{}_l \mathbb{B}^{lk} (\mathbf{Q}^T)_k{}^j = \mathbf{Q}^i{}_l \mathbf{Q}^j{}_k \mathbb{B}^{lk}$.

Transformation of quantities that involve time derivatives becomes more complicated due to the time time-dependence of $\mathbf{Q}(t)$, which follows from the assumed (in general) time-dependence of the orientation of the frame of observer $\mathcal{O}^*$ with respect to the observer $\mathcal{O}$. In this regard, we are dealing with what is sometimes called *Euclidean class of transformations*, as opposed to *Galilean transformations*, for which the $\mathbf{Q}$ does not depend on time (but $\mathbf{c}$ does but only linearly). So for instance the velocity gradient $\mathbb{L}$ and its symmetric part $\mathbb{D}$ and antisymmetric part $\mathbb{W}$ transform as follows (as you showed in the homework)

$$\mathbb{L}^* = \mathbf{Q}\mathbb{L}\mathbf{Q}^T + \dot{\mathbf{Q}}\mathbf{Q}^T \tag{2.0.7}$$

$$\mathbb{D}^* = \mathbf{Q}\mathbb{D}\mathbf{Q}^T \tag{2.0.8}$$

$$\mathbb{W}^* = \mathbf{Q}\mathbb{W}\mathbf{Q}^T + \dot{\mathbf{Q}}\mathbf{Q}^T . \tag{2.0.9}$$

where we used the antisymmetry of $\dot{\mathbf{Q}}\mathbf{Q}$ which follows from time-differentiating the orthogonality property $\mathbf{Q}(t)\mathbf{Q}^T(t) = \mathbb{I}$. Let us now introduce the following notion of *objectivity*.

Definition: scalar a, vector $\mathbf{a}$ and tensor $\mathbb{A}$ is said to be (Euclidean) **objective** if under the change of observer, it transforms as follows

$$a^* = a , \tag{2.0.10}$$

$$\mathbf{a}^* = \mathbf{Q}\mathbf{a}, \qquad \text{in the sense } (\mathbf{a}^*)^i = \mathbf{Q}^i{}_j \mathbf{a}^j , \tag{2.0.11}$$

$$\mathbb{A}^* = \mathbf{Q}\mathbb{A}\mathbf{Q}^T , \qquad \text{in the sense } (\mathbb{A}^*)^{ij} = \mathbf{Q}^i{}_k \mathbb{A}^{kl} (\mathbf{Q}^T)_l{}^j = \mathbf{Q}^i{}_k \mathbf{Q}^j{}_l \mathbb{A}^{kl} . \tag{2.0.12}$$

So for **kinematic** quantities, we can check whether or not they are objective. For instance since the transformation rule for change of observer reads $\mathbb{F}^* = \mathbf{Q}\mathbb{F}$, it is not objective tensor, but note that its individual columns transform as objective vectors. The left Cauchy Green tensor $\mathbb{C}$ is not an objective tensor (since $\mathbb{C}^* = \mathbb{C}$), but its entries are objective scalars. Finally the Right Cauchy Green tensor is an objective tensor, since $\mathbb{B}^* = \mathbf{Q}\mathbb{B}\mathbf{Q}^T$ and the same holds for the symmetric velocity gradient $\mathbb{D}$. Neither velocity gradient $\mathbb{L}$ nor its antisymmetric part $\mathbb{W}$ are objective tensors.

While for kinematic quantities as the examples above, one can verify of reject their objectivity, for other quantities in the physical theory, it must be postulated. In the context of continuum mechanics and thermodynamics of solids, it is postulated that *density ρ, internal energy e, entropy η and consequently also all other thermodynamic potentials such as the Helmholtz or Gibbs free energies or enthalpy, are objective scalars, heat flux vector $\mathbf{q}$ is an objective vector and the Cauchy stress $\mathbb{T}$ is an objective tensor*. Note that consequently, the stress vector $\mathbf{t}_{(\boldsymbol{n})} = \mathbb{T}\boldsymbol{n}$ expressing force exerted by surface forces on a unit surface with outer unit normal $\boldsymbol{n}$ is an objective vector.

Another notion of objectivity arises when we start to talk about *constitutive functions* or functionals, i.e. about constitutive theory. Property of the constitutive functions that reflects their objectivity is called *material frame indifference* and let us define it through illustrative examples. It must be noted that material frame indifference is not a physical principle really, it is more a meta-physical one, which basically reflects the requirement (that seems not to be violated in the nature) that the *constitutive functionals do not depend on the observer*. Let us make this more precise.

Material frame indifference Let us consider an objective scalar ψ (e.g. free energy), objective vector $\mathbf{t}$ (e.g. stress vector) and objective tensor $\mathbb{T}$ (e.g. Cauchy stress tensor) and let us assume that for a given material point $\mathbf{X}$ and time t, the value of these fields is given by

$$\psi(\mathbf{X},t) = \widehat{\psi}\left(\boldsymbol{\chi}(\mathbf{X}',t'), \text{s.t. } \mathbf{X}' \in \boldsymbol{\kappa}_0, t' \le t; \mathbf{X}\right) , \tag{2.0.13}$$

$$\mathbf{t}(\mathbf{X},t) = \widehat{\mathbf{t}}\left(\boldsymbol{\chi}(\mathbf{X}',t'), \text{s.t. } \mathbf{X}' \in \boldsymbol{\kappa}_0, t' \le t; \mathbf{X}\right) , \tag{2.0.14}$$

$$\mathbb{T}(\mathbf{X},t) = \widehat{\mathbb{T}}\left(\boldsymbol{\chi}(\mathbf{X}',t'), \text{s.t. } \mathbf{X}' \in \boldsymbol{\kappa}_0, t' \le t; \mathbf{X}\right) , \tag{2.0.15}$$

i.e. they are determined by constitutive functionals $\widehat{\psi}$, $\widehat{\mathbf{t}}$ and $\widehat{\mathbb{T}}$ depending on the history of motion $\boldsymbol{\chi}$ of all particles in the body until time t (causality). The last argument $\mathbf{X}$ reflects that the functionals may differ from point to point within the body (i.e. the body can be heterogeneous).

Then **principle of material indifference** asserts the following properties of $\widehat{\psi}$, $\widehat{\mathbf{t}}$, $\widehat{\mathbb{T}}$ (we simplify notation in the arguments by omitting $\mathbf{X}'\in\boldsymbol{\kappa}_0, t' \le t$):

$$\widehat{\psi}\left(\boldsymbol{\chi}(\mathbf{X}',t');\mathbf{X}\right) = \widehat{\psi}\left(\boldsymbol{\chi}^*(\mathbf{X}',t');\mathbf{X}\right), \tag{2.0.16}$$

$$\mathbf{Q}(t)\widehat{\mathbf{t}}\left(\boldsymbol{\chi}(\mathbf{X}',t');\mathbf{X}\right) = \widehat{\mathbf{t}}\left(\boldsymbol{\chi}^*(\mathbf{X}',t');\mathbf{X}\right), \tag{2.0.17}$$

$$\mathbf{Q}(t)\widehat{\mathbb{T}}\left(\boldsymbol{\chi}^*(\mathbf{X}',t');\mathbf{X}\right)\mathbf{Q}^T(t) = \widehat{\mathbb{T}}\left(\boldsymbol{\chi}^*(\mathbf{X}',t');\mathbf{X}\right), \tag{2.0.18}$$

holds for all observer transformations and all deformations. Note that $\boldsymbol{\chi}^*$ is the deformation of the body viewed by the "starred" observer. Consider now so called **simple materials**, for which the dependence on all the history of all material points reduces to dependence on history of just a local neighborhood of the material point $\mathbf{X}$ in terms of the deformation gradient $\mathbb{F}(\mathbf{X},t)$. For such materials, using that $\mathbb{F}^* = \mathbf{Q}\mathbb{F}$, the material frame indifference asserts that

$$\widehat{\psi}\left(\mathbb{F}(\mathbf{X},t');\mathbf{X}\right) = \widehat{\psi}\left(\mathbf{Q}(t')\mathbb{F}(\mathbf{X},t');\mathbf{X}\right), \tag{2.0.19}$$

$$\mathbf{Q}(t)\widehat{\mathbf{t}}\left(\mathbb{F}(\mathbf{X},t');\mathbf{X}\right) = \widehat{\mathbf{t}}\left(\mathbf{Q}(t')\mathbb{F}(\mathbf{X},t');\mathbf{X}\right), \tag{2.0.20}$$

$$\mathbf{Q}(t)\widehat{\mathbb{T}}\left(\mathbb{F}(\mathbf{X},t');\mathbf{X}\right)\mathbf{Q}^T(t) = \widehat{\mathbb{T}}\left(\mathbf{Q}(t')\mathbb{F}(\mathbf{X},t');\mathbf{X}\right). \tag{2.0.21}$$

for all histories of $\mathbf{Q}(t'), t' \le t$ and all deformations. This example of constitutive response still covers memory effects (e.g. viscoelasticity). Simplifying even more, by allowing only dependence on the actual deformation state, which is the situation characteristic for elastic solids, the constitutive functionals reduce to mere functions, for which the principle of material frame indifference takes the following form

$$\widehat{\psi}\left(\mathbb{F}(\mathbf{X},t);\mathbf{X}\right) = \widehat{\psi}\left(\mathbf{Q}(t)\mathbb{F}(\mathbf{X},t);\mathbf{X}\right), \tag{2.0.22}$$

$$\mathbf{Q}(t)\widehat{\mathbf{t}}\left(\mathbb{F}(\mathbf{X},t);\mathbf{X}\right) = \widehat{\mathbf{t}}\left(\mathbf{Q}(t)\mathbb{F}(\mathbf{X},t);\mathbf{X}\right), \tag{2.0.23}$$

$$\mathbf{Q}(t)\widehat{\mathbb{T}}\left(\mathbb{F}(\mathbf{X},t);\mathbf{X}\right)\mathbf{Q}^T(t) = \widehat{\mathbb{T}}\left(\mathbf{Q}(t)\mathbb{F}(\mathbf{X},t);\mathbf{X}\right). \tag{2.0.24}$$

holds for all $\mathbf{Q}(t) \in$ orth and all deformations. We will mostly make use of this form of the principle of material indifference in this lecture.

Change of reference configuration

Another important class of transformations that allows us to define the *material symmetry group* is the change of reference configuration. Consider reference configuration $\boldsymbol{\kappa}_0$ and a different configuration $\boldsymbol{\kappa}_0^*$ related with it by the mapping $\boldsymbol{\Lambda}$

$$\mathbf{X}^* = \boldsymbol{\Lambda}(\mathbf{X}). \tag{2.0.25}$$

The position of a material point $\mathbf{X}$ at time t in the actual configuration is given by the motion $\boldsymbol{\chi}$ as follows

$$\mathbf{x} = \boldsymbol{\chi}_{\boldsymbol{\kappa}_0}(\mathbf{X},t) = \boldsymbol{\chi}_{\boldsymbol{\kappa}_0^*}(\mathbf{X}^*,t) \tag{2.0.26}$$

Let us investigate the relationship between the corresponding deformation gradients:

$$(\mathbb{F}_{\boldsymbol{\kappa}_0})^i{}_J = \frac{\partial \chi^i_{\boldsymbol{\kappa}_0}(\mathbf{X},t)}{\partial \mathbf{X}^J} = \frac{\partial \chi^i_{\boldsymbol{\kappa}_0^*}(\boldsymbol{\Lambda}(\mathbf{X}),t)}{\partial \mathbf{X}^J} = \frac{\partial \chi^i_{\boldsymbol{\kappa}_0^*}(\mathbf{X}^*,t)}{\partial \mathbf{X}^{*K}}\frac{\partial \boldsymbol{\Lambda}^K(\mathbf{X})}{\partial \mathbf{X}^J} = (\mathbb{F}_{\boldsymbol{\kappa}_0^*})^i{}_K(\mathbf{X}^*,t)\frac{\partial \boldsymbol{\Lambda}^K(\mathbf{X})}{\partial \mathbf{X}^J}. \tag{2.0.27}$$

Defining the matrix field

$$\mathbb{P}(\mathbf{X}) \stackrel{\text{def}}{=} \mathrm{Grad}_{\mathbf{X}}\boldsymbol{\Lambda} \qquad \text{by} \quad \mathbb{P}^K{}_J(\mathbf{X}) \stackrel{\text{def}}{=} \frac{\partial \boldsymbol{\Lambda}^K(\mathbf{X})}{\partial \mathbf{X}^J}, \tag{2.0.28}$$

we arrive at the following relation between $\mathbb{F}_{\boldsymbol{\kappa}_0}$ and $\mathbb{F}_{\boldsymbol{\kappa}_0^*}$:

$$\boxed{\mathbb{F}_{\boldsymbol{\kappa}_0^*}(\mathbf{X}^*,t) = \mathbb{F}_{\boldsymbol{\kappa}_0}(\mathbf{X},t)\mathbb{P}^{-1}(\mathbf{X})} \qquad \text{in the followign sense} \quad (\mathbb{F}_{\boldsymbol{\kappa}_0^*})^i{}_J(\boldsymbol{\Lambda}(\mathbf{X}),t) = (\mathbb{F}_{\boldsymbol{\kappa}_0})^i{}_K(\mathbf{X},t)\,(\mathbb{P}^{-1})^K_J(\mathbf{X}). \tag{2.0.29}$$

Material symmetry is a material property related to a particular response function or functional. Symmetry of some property (say Cauchy stress functional) might be in principle different than symmetry of the heat flux functional for the same material. Let us consider the Cauchy stress, being interested in mechanics, and let us restrict ourselves to elastic materials with the response given by function

$$\mathbb{T}_{\boldsymbol{\kappa}_0}(\mathbf{X},t) = \widehat{\mathbb{T}}_{\boldsymbol{\kappa}_0}(\mathbb{F}(\mathbf{X},t),,\mathbf{X}), \tag{2.0.30}$$

i.e. without any memory and with only weakly-nonlocal dependence on the deformation (via the first gradient of the deformation mapping). Let us omit explicit dependence on **X** by considering *homogeneous* materials.

Let us know, that we now explicitly write the reference configuration in the subscript to differentiate between the response functionals with respect to different reference configurations. Clearly, by an alternative choice of the reference configuration $\boldsymbol{\kappa}_0^*$, we can also describe the response of the material through another response function $\widehat{\mathbb{T}}_{\boldsymbol{\kappa}_0^*}$, and the two are related via

$$\widehat{\mathbb{T}}_{\boldsymbol{\kappa}_0^*}(\mathbb{F}_{\boldsymbol{\kappa}_0^*}(\mathbf{X}^*,t)) = \widehat{\mathbb{T}}_{\boldsymbol{\kappa}_0}(\mathbb{F}_{\boldsymbol{\kappa}_0}(\mathbf{X},t)), \tag{2.0.31}$$

where $\mathbf{X}^* = \boldsymbol{\Lambda}(\mathbf{X})$.

Definition: We say that that a matrix $\mathbb{P}$ (defined as Grad$\boldsymbol{\Lambda}$) is in the **local material symmetry group** $\mathcal{G}$ of the functional $\widehat{\mathbb{T}}_{\boldsymbol{\kappa}_0}$ at point **X**, if it holds

$$\widehat{\mathbb{T}}_{\boldsymbol{\kappa}_0}() = \widehat{\mathbb{T}}_{\boldsymbol{\kappa}_0^*}(),, \tag{2.0.32}$$

where $\boldsymbol{\kappa}_0$ and $\boldsymbol{\kappa}_0^*$ are related by the mapping Λ and the brackets () take the **same** arguments.

This may be difficult to interpret at fist glance, so let us write down an equivalent characterization in terms of one of the functionals. Using (2.0.31), we get

$$\widehat{\mathbb{T}}_{\boldsymbol{\kappa}_0}(\mathbb{F}_{\boldsymbol{\kappa}_0}(\mathbf{X},t)) = \widehat{\mathbb{T}}_{\boldsymbol{\kappa}_0^*}(\mathbb{F}_{\boldsymbol{\kappa}_0^*}(\mathbf{X}^*,t)) = \widehat{\mathbb{T}}_{\boldsymbol{\kappa}_0}(\mathbb{F}_{\boldsymbol{\kappa}_0^*}(\mathbf{X}^*,t)) = \widehat{\mathbb{T}}_{\boldsymbol{\kappa}_0}(\mathbb{F}_{\boldsymbol{\kappa}_0}(\mathbf{X},t)\mathbb{P}^{-1}), \tag{2.0.33}$$

or, in short, if

$$\boxed{\widehat{\mathbb{T}}_{\boldsymbol{\kappa}_0}(\mathbb{F}_{\boldsymbol{\kappa}_0}(\mathbf{X},t)) = \widehat{\mathbb{T}}_{\boldsymbol{\kappa}_0}(\mathbb{F}_{\boldsymbol{\kappa}_0}(\mathbf{X},t)\mathbb{P}^{-1}) \quad \text{for all} \quad \mathbb{F}_{\boldsymbol{\kappa}_0}} \tag{2.0.34}$$

This is often abbreviated even more as

$$\boxed{\widehat{\mathbb{T}}(\mathbb{F}\mathbb{P}) = \widehat{\mathbb{T}}(\mathbb{F})} \quad \text{for all} \quad \mathbb{F} \tag{2.0.35}$$

(obtained by replacing without loss of generality $\mathbb{F}$ by $\mathbb{F}\mathbb{P}$ in the former relation).

An interpretation useful for visualization of the above expression is as follows: the response of the material to given deformation does not change, if we first apply another deformation $\mathbb{P}$ in the reference configuration (for instance some particular rotation). This idea probably best coincides with our intuitive notion how the material symmetry should manifest itself in the constitutive response.