Homework nr. 2 - Material symmetry - deadline 9.4.

During the last exercises we discussed another important class of transformations - the change of reference configuration - using which we defined for a given constitutive relation (e.g. Cauchy stress) the symmetry group of the material. Let us recall briefly what we did. Let us consider two reference configurations $\kappa(\mathcal{B})$ and $\kappa^*(\mathcal{B})$ related through a (sufficiently smooth as always) bijestion $\mathbf{\Lambda} : \kappa(\mathcal{B}) \to \kappa^*(\mathcal{B})$:

$$\mathbf{X}^\star = \mathbf{\Lambda}(\mathbf{X}) \, .$$

The deformation gradient of any deformation can then be expressed with respect to both of these reference configurations and the two deformation gradients are related as follows

$$\mathbb{F}_\kappa = \mathbb{F}_{\kappa^*}\mathbb{P} \qquad\qquad (\mathbb{F}_\kappa)^i{}_J = (\mathbb{F}_{\kappa^*})^i{}_K \mathbb{P}^K_{\ J},$$

where

$$\mathbb{P}^K_{\ J} := \frac{\partial \mathbf{\Lambda}^K}{\partial X^J} \, .$$

1. Examine the transformation properties under the change of reference configuration of Cartesian components of kinematic tensors of continuum mechanics: the deformation gradient $\mathbb{F}$, Right Cauchy-Green tensor $\mathbb{C} = \mathbb{F}^T\mathbb{F}$, Left Cauchy-Green tensor $\mathbb{B} = \mathbb{F}\mathbb{F}^T$, velocity gradient $\mathbb{L} = \mathrm{grad}\mathbf{v}$, symmetric part of the velocity gradient $\mathbb{D} = \mathrm{sym}(\mathbb{L})$, antisymmetric part of the velocity gradient $\mathbb{W} = \mathrm{skew}(\mathbb{L})$, Green-Saint Venant tensor $\mathbb{E} = \frac{1}{2}(\mathbb{C} - \mathbb{I})$, Euler-Almansi tensor $= \frac{1}{2}(\mathbb{I} - \mathbb{B}^{-1})$ and the relative deformation gradient $\mathbb{F}_t(\mathbf{x}, \tau) = \mathbb{F}(\mathbf{X}, \tau)\mathbb{F}^{-\mathbf{1}}(\mathbf{X}, t)|_{\mathbf{X}=\boldsymbol{\chi}^{-\mathbf{1}}(\mathbf{x},t)}$

2. Consider for simplicity a homogeneous elastic material. For the Cauchy stress tensor $\mathbb{T} = \hat{\mathbb{T}}_\kappa(\mathbb{F}_\kappa(\mathbf{X}, t))$, we define the material symmetry group $\mathcal{G}_\kappa \subset \mathrm{unim}_+$, where $\mathrm{unim}_+ := \{\mathbb{P} \in \mathbb{R}^{3\times3}, \det\mathbb{P} = 1\}$ by the relation

$$\mathbb{P} \in \mathcal{G}_\kappa \Longleftrightarrow \forall \mathbb{F}_\kappa \in \{\mathbb{F}_\kappa \in \mathbb{R}^{3\times3}, \det\mathbb{F}_\kappa > 0\} : \hat{\mathbb{T}}_\kappa(\mathbb{F}_\kappa) = \hat{\mathbb{T}}_\kappa(\mathbb{F}_\kappa\mathbb{P}) \qquad (1)$$

   (a) Show that $\mathcal{G}_\kappa$ is a **group** (with respect to the group binary operation being the composition of the mappings, i.e. considering $\forall \mathbb{P}_1, \mathbb{P}_2 \in \mathcal{G}_\kappa, \mathbb{P}_1 \cdot \mathbb{P}_2 := \mathbb{P}_1\mathbb{P}_2$).

   (b) Prove (for the Cauchy stress tensor $\hat{\mathbb{T}}_\kappa(\mathbb{F}_\kappa)$) *Noll's rule*: Let $\mathcal{G}_\kappa$ be the symmetry group of a material with respect to reference configuration $\boldsymbol{\kappa}(\mathcal{B})$, then with respect to reference configuration $\boldsymbol{\kappa}^*(\mathcal{B}) := \mathbf{\Lambda}(\boldsymbol{\kappa}(\mathcal{B}))$ the corresponding symmetry group reads $\mathcal{G}_{\kappa^*} = \mathbb{Q}\mathcal{G}_\kappa\mathbb{Q}^{-1} := \{\mathbb{Q}\mathbb{P}\mathbb{Q}^{-1}; \mathbb{P} \in \mathcal{G}_\kappa\}$, where $\mathbb{Q}^I{}_J := \frac{\partial \mathbf{\Lambda}^I}{\partial X^J}$ (and we have shown that this implies relation for transformation of the deformation gradient $\mathbb{F}_{\kappa^*} = \mathbb{F}_\kappa\mathbb{Q}^{-1}$). HINT: Make use of the relation $\hat{\mathbb{T}}_\kappa(\mathbb{F}_\kappa) = \hat{\mathbb{T}}_{\kappa^*}(\mathbb{F}_{\kappa^*})$, by which we defined $\hat{\mathbb{T}}_{\kappa^*}$.

   (c) Express the corresponding condition of material symmetry for the second Piola-Kirchhoff stress tensor.