Chapter 4

4.1 Anisotropy

We have shown last time that for an *isotropic* hyperelastic solid, the free energy is an objective scalar of one (symmetric) tensorial variable (e.g. $\mathbb{C}$ or $\mathbb{B}$), which implies its representation - it is only a function of the three principal invariants of the tensorial argument. Consequently, we obtained representation for all required stress measures (Cauchy stress, first and second Piola-Kirchhoff stresses). The assumption of material isotropy - biggest possible material symmetry group for solids - is, however, very restraining. What if the symmetry of the material is lower, is there any chance to get representation as well? The answer is yes, but we need to generalize the representation theorem.

4.1.1 Representation theorem for isotropic functions of more tensorial arguments

So far, we learned a representation for an scalar isotropic function of one symmetric tensorial (2nd order argument). We will need to extend this to the following setting.

Definition: We say that a scalar function $\widehat{W}(\mathbb{A}_i)$ of 2 symmetric tensorial arguments $\mathbb{A}_1$, $\mathbb{A}_2$, is isotropic if

$$\widehat{W}(\mathbf{Q}\mathbb{A}_1\mathbf{Q}^T, \mathbf{Q}\mathbb{A}_2\mathbf{Q}^T) = \widehat{W}(\mathbb{A}_1, \mathbb{A}_2) \qquad \forall \mathbf{Q} \in \mathrm{orth}\ . \tag{4.1.1}$$

The following representation theorem holds (Smith, 1965):

Theorem 4.1.1 (Representation theorem for isotropic functions of 2 symmetric tensorial arguments)**.** *Let $\widehat{W}(\mathbb{A}_1, \mathbb{A}_2)$, be an isotropic function of 2 symmetric second-order tensorial arguments $\mathbb{A}_1$, $\mathbb{A}_2$. Then it is a function of the following invariants:*

- $\mathrm{tr}\,\mathbb{A}_i$, $\mathrm{tr}\,\mathbb{A}_i^2$, $\mathrm{tr}\,\mathbb{A}_i^3$, $\qquad i = 1,2$
- $\mathrm{tr}(\mathbb{A}_i\mathbb{A}_j)$, $\mathrm{tr}(\mathbb{A}_i^2\mathbb{A}_j)$, $\mathrm{tr}(\mathbb{A}_i\mathbb{A}_j^2)$, $\mathrm{tr}(\mathbb{A}_i^2\mathbb{A}_j^2)$, $\qquad i,j = 1,2,\ i \neq j$

Remark 1. Note that the first line are invariant of single arguments, the second line are coupled invariants of pairs of tensors.

Exercise 6. The first line are just invariants of single tensorial arguments. For $N = 1$, we should get equivalent representation as before. Is that so? HINT: Use Cayley-Hamilton theorem.
Solution: We want to show, that we can express invariants $\mathrm{tr}\,\mathbb{A}_i$, $\mathrm{tr}\,\mathbb{A}_i^2$, $\mathrm{tr}\,\mathbb{A}_i^3$ is terms of the principal invariants. But the first one is already $I_1(\mathbb{A})$, the second one

$$\mathrm{tr}\,\mathbb{A}^2 = I_1^2(\mathbb{A}) - 2I_2(\mathbb{A})\ , \tag{4.1.2}$$

and since Cayley-Hamilton theorem tells us that:

$$-\mathbb{A}^3 + I_1(\mathbb{A})\mathbb{A}^2 - I_2(\mathbb{A})\mathbb{A} + I_3(\mathbb{A})\mathbb{I} = \mathbf{0}\ , \tag{4.1.3}$$

taking a trace gives us

$$\begin{aligned} \mathrm{tr}\,\mathbb{A}^3 &= I_1(\mathbb{A})\,\mathrm{tr}\,\mathbb{A}^2 - I_2(\mathbb{A})\,\mathrm{tr}\,\mathbb{A} + 3I_3(\mathbb{A}) \\ &= I_1(\mathbb{A})(I_1^2(\mathbb{A}) - 2I_2(\mathbb{A})) - I_2(\mathbb{A})I_1(\mathbb{A}) + 3I_3(\mathbb{A})\ , \end{aligned} \tag{4.1.4}$$

Similarly, we get the inverse:

$$I_1(\mathbb{A}) = \operatorname{tr}\mathbb{A} \ , \tag{4.1.5}$$

$$I_2(\mathbb{A}) = \frac{1}{2}\left((\operatorname{tr}\mathbb{A})^2 - \operatorname{tr}\mathbb{A}^2\right)z, \tag{4.1.6}$$

$$I_3(\mathbb{A}) = \frac{1}{3}\left(\operatorname{tr}\mathbb{A}^3 - \operatorname{tr}\mathbb{A}\operatorname{tr}\mathbb{A}^2 + \frac{1}{2}((\operatorname{tr}\mathbb{A})^2 - \operatorname{tr}\mathbb{A}^2)\operatorname{tr}\mathbb{A}\right) \ , \tag{4.1.7}$$

so indeed, we can equivalently use the three independent functions $\operatorname{tr}\mathbb{A}$, $\operatorname{tr}\mathbb{A}^2$, $\operatorname{tr}\mathbb{A}^3$ as and equivalent set of representing invariants, alternatively to the principle invariants.

4.1.2 Transversally anisotropic hyperelastic material

If we cannot assume material isotropy, we must in our search for representation retrace our steps back a little bit. For the free energy of a hyperelastic solid, we know $W = \hat{W}(\mathbb{F})$ and using the principle of material frame indifference, we were able to obtain a reduced representation of the form $W = \widetilde{W}(\mathbb{C})$.

For an anisotropic solid material, we already know that it must hold

$$\widetilde{W}(\mathbf{Q}\mathbb{C}\mathbf{Q}^T) = \widetilde{W}(\mathbb{C}) \qquad \forall \mathbf{Q} \in \mathscr{G} \subset \text{orth} \ , \tag{4.1.8}$$

for symmetry group $\mathscr{G}$ - a (proper) subgroup of the orthogonal group. For general $\mathscr{G}$, this is not enough information to provide representation, but in some cases, it is possible to do so. Let us examine the simplest anisotropic material, so called *transversally anisotropic* material. Such materials have (locally) some specific direction (symmetry axis) ℓ. The material has different response in the direction along ℓ and different in planes perpendicular to ℓ. Within such planes, the material is however already isotropic. Examples can be materials composed of aligned fibers, then the direction of these fibers determine ℓ.

Based on the description of the isotropy group for a transversally anisotropic material, we can try to formulate mathematically, what is the corresponding symmetry group $\mathscr{G}$. Given the orientation of the axis by a unit vector ℓ, the group clearly must be set of all rotations around this axis plus reflections with respect to planes perpendicular to the axis.

Exercise 7. How do such rotations look like, or, in other words, what is the rotation matrix for a rotation around given axis by a given angle?
Solution: Answer can be found easily by drawing a picture and realizing we know the rotation matrix in 2D.

$$\mathbf{Q}_\ell(\vartheta) = \cos\vartheta\mathbb{I} + \sin\vartheta\ell\times + (1-\cos\vartheta)\ell\otimes\ell \ , \tag{4.1.9}$$

$$(\mathbf{Q}_\ell(\vartheta))_{ik} = \cos\vartheta\delta_{ik} + \sin\vartheta\varepsilon_{ijk}\ell_j + (1-\cos\vartheta)\ell_i\ell_k \ . \tag{4.1.10}$$

All such $\mathbf{Q}_\ell$ satisfy $\mathbf{Q}_\ell\ell = \ell$, and if we add also the possibility of reflection, we get

$$\mathscr{G}^{TA} = \{\mathbf{Q} \in \text{orth} : \mathbf{Q}\ell = \pm\ell\} \ .$$

This group can be characterized nicely using the so-called *structural tensor*

$$\mathbb{S} \overset{\text{def}}{=} \ell\otimes\ell \ , \tag{4.1.11}$$

as follows:

$$\boxed{\mathscr{G}^{TA} = \{\mathbf{Q} \in \text{orth} : \mathbf{Q}\mathbb{S}\mathbf{Q}^T = \mathbb{S}\}} \ . \tag{4.1.12}$$

This lengthy detour had the following purpose. The class of symmetry groups, which can be characterized by structural tensors as in (4.1.12) admit characterization using the following

Theorem 4.1.2. *A scalar-valued function $f(\mathbb{A}_i)$ (of N symmetric 2nd order tensorial arguments) is invariant with respect to the symmetry group characterized by second-order tensors $\mathbb{S}_j$: $\mathscr{G} = \{\mathbf{Q} \in \text{orth} : \mathbf{Q}\mathbb{S}_j\mathbf{Q}^T = \mathbb{S}_j, j = 1,\ldots,M\}$, if and only if there exists an* **isotropic** *function $\tilde{f}(\mathbb{A}_i,\mathbb{S}_j)$ such that $\tilde{f}(\mathbb{A}_i,\mathbb{S}_j) = f(\mathbb{A}_i)$.*

In view of transversally anisotropic hyperelastic materials, the theorem tells us, that the free energy of such a material can be written as

$$W = \widehat{W}(\mathbb{C}, \mathbb{S}) , \tag{4.1.13}$$

where $\widehat{W}$ is **isotropic** function (as a function of two tensorial arguments). But now, we can employ the representation theorem 4.1.1, to obtain its representation.

Exercise 8. Use representation theorem 4.1.1 to deduce the representation for free energy of transversally anisotropic hyperelastic material.
Solution: So the theorem tells us, that the representation can be found in terms of the following invariants:

- $\operatorname{tr}\mathbb{C}, \operatorname{tr}\mathbb{C}^2, \operatorname{tr}\mathbb{C}^3, \operatorname{tr}\mathbb{S}, \operatorname{tr}\mathbb{S}^2, \operatorname{tr}\mathbb{S}^3$
- $\operatorname{tr}\mathbb{C}\mathbb{S}$, $\operatorname{tr}\mathbb{C}^2\mathbb{S}$, $\operatorname{tr}\mathbb{C}\mathbb{S}^2$, $\operatorname{tr}\mathbb{C}^2\mathbb{S}^2$

and that's it (no triplets available). But

$$\operatorname{tr}\mathbb{S} = \operatorname{tr}(\ell \otimes \ell) = \ell \cdot \ell = 1 , \tag{4.1.14}$$

and also $\mathbb{S}^2 = (\ell \otimes \ell)(\ell \otimes \ell) = \ell \otimes (\ell \cdot \ell)\ell = \ell \otimes \ell$ and similarly for $\mathbb{S}^3 = \ell \otimes \ell$, so we can ignore the three invariants $\operatorname{tr}\mathbb{S}$, $\operatorname{tr}\mathbb{S}^2$, $\operatorname{tr}\mathbb{S}^3$ as they are trivial. Finally, having $\ell^2 = \ell$ implies that there are only two new coupled invariant:

$$\operatorname{tr}(\mathbb{C}\mathbb{S}) = \operatorname{tr}(\mathbb{C}(\ell \otimes \ell)) = \ell \cdot \mathbb{C}\ell , \tag{4.1.15}$$

$$\operatorname{tr}(\mathbb{C}^2\mathbb{S}) = \operatorname{tr}(\mathbb{C}^2(\ell \otimes \ell)) = \ell \cdot \mathbb{C}^2\ell . \tag{4.1.16}$$

Replacing again $\operatorname{tr}C$, $\operatorname{tr}\mathbb{C}^2$, $\operatorname{tr}\mathbb{C}^3$ by the three principal invariants, we obtain the final representation for the free energy of a transversally anisotropic hyperelastic material with axis ℓ in the form:

$$\boxed{W^{TA} = \overline{W}^{TA}(I_1(\mathbb{C}), I_2(\mathbb{C}), I_3(\mathbb{C}), \ell \cdot \mathbb{C}\ell, \ell \cdot \mathbb{C}^2\ell)} . \tag{4.1.17}$$

Exercise 9 (Homework). Using the results of the previous homework, derive the general form for $\mathbb{T}$, $\mathbb{T}^{(1)}$ and $\mathbb{T}^{(2)}$ for a hyperelastic transversally anisotropic solid with anisotropy axis ℓ.