Chapter 5

5.1 Several examples of hyperelatic materials

Let us now give several important examples of isotropic hyperelastic materials that are used in engineering and other applications.

- **The Mooney-Rivlin model (compressible)** - the free energy function reads

$$W(\mathbb{B}) = c(J-1)^2 - 2(c_1+c_2)\ln J + c_1(I_1(\mathbb{B})-3) + c_2(I_2(\mathbb{B})-3)\ , \tag{5.1.1}$$

where $J \stackrel{\text{def}}{=} \sqrt{I_3(\mathbb{B})} = \sqrt{\det \mathbb{B}}$ and c, c_1 and c_2 are constant parameters.

Exercise 10 (Homework). Show that the constitutive equation for the Cauchy stress $\mathbb{T}$ for the compressible Mooney-Rivlin material reads:

$$\mathbb{T}(\mathbb{B}) = \frac{2}{J}\left\{(cJ(J-1) - (c_1+c_2))\mathbb{I} + (c_1 + c_2 I_1(\mathbb{B}))\mathbb{B} - c_2\mathbb{B}^2\right\}\ . \tag{5.1.2}$$

- **The Neo-Hookean Model (compressible)**

$$W(\mathbb{B}) = \frac{\lambda}{2}(\ln J)^2 - \mu \ln J + \frac{\mu}{2}(I_1(\mathbb{B}) - 3)\ , \tag{5.1.3}$$

where λ and μ are Lame parameters (bulk and shear modulus, respectively).

Exercise 11 (Homework). Show, that the constitutive equation for the Cauchy stress for a compressible neo-Hookean material reads

$$\mathbb{T}(\mathbb{B}) = \frac{1}{J}\left(\lambda \ln J \mathbb{I} + \mu(\mathbb{B} - \mathbb{I})\right). \tag{5.1.4}$$

Exercise 12. Perform geometric linearization of the above constitutive equation and show that you obtain classical isotropic Hooke's law.

Solution: By geometric linearization, we mean that we expand

$$\mathbb{F} = \mathbb{I} + \nabla \mathbf{U} \tag{5.1.5}$$

$$\mathbb{B} = \mathbb{F}\mathbb{F}^T = (\mathbb{I} + \nabla \mathbf{U})(\mathbb{I} + (\nabla \mathbf{U})^T) = \mathbb{I} + 2\mathbb{H} + O(|\nabla U|^2)\ , \tag{5.1.6}$$

with $\mathbb{H} \stackrel{\text{def}}{=} \frac{1}{2}(\nabla \mathbf{U} + (\nabla \mathbf{U})^T)$. Now,

$$J = \sqrt{\det \mathbb{B}} = \sqrt{\det(\mathbb{I} + 2\mathbb{H} + O(|\nabla \mathbf{U}|^2))} = \sqrt{1 + 2\,\mathrm{tr}\,\mathbb{H} + O(|\nabla \mathbf{U}|^2)} = 1 + \mathrm{tr}\,\mathbb{H} + O(|\nabla \mathbf{U}|^2)\ , \tag{5.1.7}$$

and thus

$$\ln J = \ln(1 + \mathrm{tr}\,\mathbb{H} + O(|\nabla \mathbf{U}|^2)) = \mathrm{tr}\,\mathbb{H} + O(|\nabla \mathbf{U}|^2)\ . \tag{5.1.8}$$

Consequently, we obtain

$$\begin{aligned}\mathbb{T} &= \frac{1}{1 + \mathrm{tr}\,\mathbb{H} + O(|\nabla \mathbf{U}|^2)}\left(\lambda(\mathrm{tr}\,\mathbb{H} + O(|\nabla \mathbf{U}|^2))\mathbb{I} + 2\mu\mathbb{H}\right) \\ &= \lambda(\mathrm{tr}\,\mathbb{H})\mathbb{I} + 2\mu\mathbb{H} + O(|\nabla \mathbf{U}|^2)\ .\end{aligned} \tag{5.1.9}$$

Considering small deformations, the (Lagrangian) tensor $\mathbb{H}$ equals the (Eulerian) small strain tensor $\boldsymbol{\varepsilon}$:

$$\mathbb{H}(\mathbf{X},t) = \frac{1}{2}(\nabla_{\mathbf{X}}\mathbf{U}(\mathbf{X},t) + (\nabla_{\mathbf{X}}\mathbf{U}(\mathbf{X},t))^T) \simeq \frac{1}{2}\left(\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{x},t) + (\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{x},t))^T\right) = \boldsymbol{\varepsilon}(\mathbf{x},t) \,, \tag{5.1.10}$$

and we get, neglecting the quadratic terms

$$\mathbb{T}(\boldsymbol{\varepsilon}) = \lambda \operatorname{tr}\boldsymbol{\varepsilon}\mathbb{I} + 2\mu\boldsymbol{\varepsilon} = \frac{3\lambda + 2\mu}{3}(\operatorname{tr}\boldsymbol{\varepsilon})\mathbb{I} + 2\mu\boldsymbol{\varepsilon}^d \,, \tag{5.1.11}$$

which is linear Hooke's law.

- **The Ogden Model (compressible)**

$$W(\lambda_1,\lambda_2,\lambda_3) = \sum_{k=1}^{N} \frac{\mu_k}{\alpha_k}\left(\lambda_1^{\alpha_k} + \lambda_2^{\alpha_k} + \lambda_3^{\alpha_k} - 3\right) \,, \tag{5.1.12}$$

where λ_i are the principal stretches (eigenvalues of $\mathbb{V}$ and $\mathbb{U}$), α_k are real constants and μ_k are (constant) shear moduli. You have already shown in exercise (5) that the corresponding Cauchy stress reads

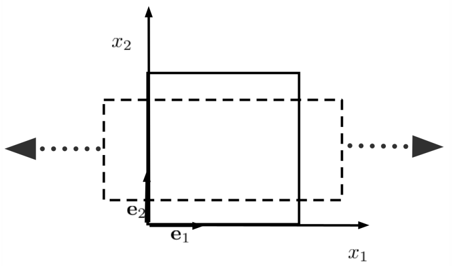
$$\mathbb{T}(\mathbb{B}) = \frac{1}{\lambda_1\lambda_2\lambda_3}\sum_{j=1}^{3}\sum_{k=1}^{N}\mu_k\lambda_j^{\alpha_k}\mathbf{g}_j^{\mathbb{B}} \otimes \mathbf{g}_j^{\mathbb{B}} \,, \tag{5.1.13}$$

where $\{\mathbf{g}_j^{\mathbb{B}}\}_{j=1}^{3}$ are eigenvectors of $\mathbb{B}$.

5.2 Examples of deformations

5.2.1 Uni-axial tension of a hyperelastic block

Consider a uni-axial tension of an isotropic hyperelastic material and let us show, how this experiment can be used to determine the constitutive parameters of a hyperelastic body. This deformation is caused by



homogeneous tensional traction force $\mathbb{T}_{11}$ applied on the two opposing faces with normals $\boldsymbol{e}_1$ and $-\boldsymbol{e}_1$. The other faces are assumed to be stress free. The deformation is expected to be homogeneous and it is just stretch, so in the Cartesian basis $\{\boldsymbol{e}_i\}_{i=1}^{3}$ the deformation gradient reads:

$$\mathbb{F} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \Longrightarrow \mathbb{B} = \mathbb{F}\mathbb{F}^T = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix} . \tag{5.2.1}$$

As we already know, by representation the general expression for Cauchy stress of an isotropic hyperelastic material can be written as

$$\mathbb{T} = \alpha_0\mathbb{I} + \alpha_1\mathbb{B} + \alpha_2\mathbb{B}^2 \,, \tag{5.2.2}$$

with α_i functions of the principal invariants. Under the assumption of purely normal loading and homogeneous deformation, the structure of the Cauchy stress is particularly simple, it has only one non-zero component - $\mathbb{T}_{11}$. So (5.2.2) reads

$$\begin{pmatrix} \mathbb{T}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \alpha_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \alpha_1 \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix} + \alpha_2 \begin{pmatrix} \lambda_1^4 & 0 & 0 \\ 0 & \lambda_2^4 & 0 \\ 0 & 0 & \lambda_3^4 \end{pmatrix} , \tag{5.2.3}$$

so the three non-trivial equations read

$$\mathbb{T}_{11} = \alpha_0 + \alpha_1 \lambda_1^2 + \alpha_2 \lambda_1^4 , \tag{5.2.4}$$

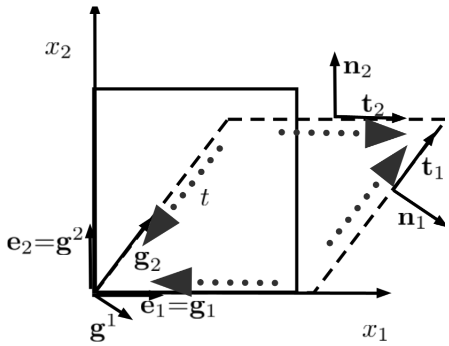
$$0 = \alpha_0 + \alpha_1 \lambda_2^2 + \alpha_2 \lambda_2^4 , \tag{5.2.5}$$

$$0 = \alpha_0 + \alpha_1 \lambda_3^2 + \alpha_2 \lambda_3^4 . \tag{5.2.6}$$

The coefficients α_i are still functions of λ_j, but for any particular form of free energy function W, their functional form is given. Measurements of stretches λ_j for various values of applied stress $\mathbb{T}_{11}$ allow (at least in principle) to determine the unknown parameters.

5.2.2 Shear of an isotropic hyperelastic square block

Let us consider another example of particular nontrivial deformation of a general isotropic hyperelastic material, in particular shear - a combination of simple shear with stretch. We will investigate the response of isotropic hyperelastic materials to such a deformation.



The deformation is caused by purely tangential (with respect to the deformed state) forces, constant along each face of the block. Clearly, one can assume that the deformation is homogeneous, in which case the deformation can be written as follows:

$$\chi_1 = \lambda_1 X_1 + k\lambda_2 X_2 , \tag{5.2.7a}$$

$$\chi_2 = \lambda_2 X_2 , \tag{5.2.7b}$$

$$\chi_3 = \lambda_3 X_3 . \tag{5.2.7c}$$

The associated deformation gradient reads

$$\mathbb{F} = \nabla_{\mathbf{X}} \boldsymbol{\chi} = \begin{pmatrix} \lambda_1 & k\lambda_2 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \Longrightarrow \mathbb{B} = \mathbb{F}\mathbb{F}^T = \begin{pmatrix} \lambda_1 & k\lambda_2 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ k\lambda_2 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 + k^2\lambda_2^2 & k\lambda_2^2 & 0 \\ k\lambda_2^2 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix} . \tag{5.2.8}$$

This matrix is block-diagonal and the upper 2×2 matrix has determinant $(\lambda_1^2 + k^2\lambda_2^2)\lambda_2^2 - k^2\lambda_2^4 = \lambda_1^2\lambda_2^2$, we can easily invert

$$\mathbb{B}^{-1} = \begin{pmatrix} \lambda_1^{-2} & -k\lambda_1^{-2} & 0 \\ -k\lambda_1^{-2} & \lambda_2^{-2} + k^2\lambda_1^{-2} & 0 \\ 0 & 0 & \lambda_3^{-2} \end{pmatrix} . \tag{5.2.9}$$

The general form of Cauchy stress for an isotropic hyperelastic body can be written as

$$\mathbb{T} = \beta_0 \mathbb{I} + \beta_1 \mathbb{B} + \beta_{-1} \mathbb{B}^{-1} \ , \tag{5.2.10}$$

where β_i are functions of the principal invariants of $\mathbb{B}$. Indeed this representation follows from the "standard" one ($\alpha_0 \mathbb{I} + \alpha_1 \mathbb{B} + \alpha_2 \mathbb{B}^2$) by eliminating $\mathbb{B}^2$ from Cayley-Hamilton theorem for $\mathbb{B}$ multiplied by $\mathbb{B}^{-1}$. Plugging in the expressions for $\mathbb{B}$ and $\mathbb{B}^{-1}$, we obtain

$$\mathbb{T} = \begin{pmatrix} \beta_0 + \beta_1(\lambda_1^2 + k^2\lambda_2^2) + \beta_{-1}\lambda_1^{-2} & \beta_1 k \lambda_2^2 - \beta_{-1} k \lambda_1^{-2} & 0 \\ \beta_1 k \lambda_2^2 - \beta_{-1} k \lambda_1^{-2} & \beta_0 + \beta_1 \lambda_2^2 + \beta_{-1}(\lambda_2^{-2} + k^2 \lambda_1^{-2}) & 0 \\ 0 & 0 & \beta_0 + \beta_1 \lambda_3^2 + \beta_{-1} \lambda_3^{-2} \end{pmatrix} . \tag{5.2.11}$$

Note that we obtained in particular that

$$\mathbb{T}_{13} = \mathbb{T}_{31} = \mathbb{T}_{23} = \mathbb{T}_{32} = 0 \ , \tag{5.2.12}$$

i.e. the material does not support any shear stress in the direction out of the $x_1 - x_2$ plane, which is hardly surprising because of the assumed deformation. We also note that the shear stress within the $x_1 - x_2$ plane is

$$\mathbb{T}_{12} = \beta_1 k \lambda_2^2 - \beta_{-1} k \lambda_1^{-2} \ , \tag{5.2.13}$$

and the following mean normal stress difference holds

$$\mathbb{T}_{11} - \mathbb{T}_{22} = \frac{\lambda_1^2 - \lambda_2^2 + k^2 \lambda_2^2}{k \lambda_2^2} \mathbb{T}_{12} \ . \tag{5.2.14}$$

In order to proceed further, we will want to employ the stress boundary conditions, which relate to the deformed state and apply on the boundaries, but due to the assumed homogeneity of the deformation, they hold everywhere. The conditions are

$$\boldsymbol{n}_1 \cdot \mathbb{T} \boldsymbol{n}_1 = 0 \ , \tag{5.2.15a}$$

$$\boldsymbol{n}_2 \cdot \mathbb{T} \boldsymbol{n}_2 = 0 \ , \tag{5.2.15b}$$

$$\mathbf{t}_1 \cdot \mathbb{T} \boldsymbol{n}_1 = t \ , \tag{5.2.15c}$$

$$\mathbf{t}_2 \cdot \mathbb{T} \boldsymbol{n}_2 = t \ . \tag{5.2.15d}$$

These four conditions express the purely tangential stress forcing in the deformed state, by tangent force with amplitude t acting along the deformed faces of the block (dotted arrows in Figure). In addition, let us assume no normal stress acting in the perpendicular direction, i.e.

$$\boldsymbol{n}_3 \cdot \mathbb{T} \boldsymbol{n}_3 = 0 \ . \tag{5.2.16a}$$

It is instructive to consider the following covariant and contravariant bases

$$\mathbf{g}_1 = \boldsymbol{e}_1 \ , \qquad \mathbf{g}^1 = \boldsymbol{e}_1 - k \boldsymbol{e}_2 \ , \tag{5.2.17}$$

$$\mathbf{g}_2 = k \boldsymbol{e}_1 + \boldsymbol{e}_2 \ , \qquad \mathbf{g}^2 = \boldsymbol{e}_2 \ , \tag{5.2.18}$$

$$\mathbf{g}_3 = \boldsymbol{e}_3 \ , \qquad \mathbf{g}^3 = \boldsymbol{e}_3 \ . \tag{5.2.19}$$

which yield immediately

$$\boldsymbol{n}_1 = \frac{\mathbf{g}^1}{\|\mathbf{g}^1\|} = \frac{\boldsymbol{e}_1 - k \boldsymbol{e}_2}{\sqrt{1+k^2}} \ , \qquad \boldsymbol{n}_2 = \frac{\mathbf{g}^2}{\|\mathbf{g}^2\|} = \boldsymbol{e}_2 \ , \tag{5.2.20}$$

$$\mathbf{t}_1 = \frac{\mathbf{g}_2}{\|\mathbf{g}_2\|} = \frac{k \boldsymbol{e}_1 + \boldsymbol{e}_2}{\sqrt{1+k^2}} \ , \qquad \mathbf{t}_2 = \frac{\mathbf{g}_1}{\|\mathbf{g}_1\|} = \boldsymbol{e}_1 \ . \tag{5.2.21}$$

The four conditions (5.2.15) then read as follows:

$$\frac{\mathbb{T}_{11} + k^2 \mathbb{T}_{22} - 2k \mathbb{T}_{12}}{1+k^2} = 0 \ , \tag{5.2.22}$$

$$\mathbb{T}_{22} = 0 \ , \tag{5.2.23}$$

$$\frac{k \mathbb{T}_{11} - k \mathbb{T}_{22} + (1-k^2) \mathbb{T}_{12}}{1+k^2} = t \ , \tag{5.2.24}$$

$$\mathbb{T}_{12} = t \ , \tag{5.2.25}$$

which imply

$$\mathbb{T}_{11} = 2k\mathbb{T}_{12}\ , \qquad \mathbb{T}_{22} = 0\ , \qquad \mathbb{T}_{12} = t\ . \tag{5.2.26}$$

Using also (5.2.14), which now reads (since $\mathbb{T}_{22} = 0$))

$$\mathbb{T}_{11} = \frac{\lambda_1^2 + (1-k^2)\lambda_2^2}{k\lambda_2^2}\mathbb{T}_{12}\ , \tag{5.2.27}$$

we arrive at

$$\lambda_1^2 = \lambda_2^2(1+k^2)\ . \tag{5.2.28}$$

The Cauchy stress can therefore be rewritten as follows

$$\mathbb{T} = \mathbb{T}_{11}\boldsymbol{e}_1\otimes\boldsymbol{e}_1 + \mathbb{T}_{12}(\boldsymbol{e}_1\otimes\boldsymbol{e}_2 + \boldsymbol{e}_2\otimes\boldsymbol{e}_1) + \mathbb{T}_{33}\boldsymbol{e}_3\otimes\boldsymbol{e}_3 \tag{5.2.29}$$

$$= \mathbb{T}_{12}(2k\boldsymbol{e}_1\otimes\boldsymbol{e}_1 + \boldsymbol{e}_1\otimes\boldsymbol{e}_2 + \boldsymbol{e}_2\otimes\boldsymbol{e}_1) + \mathbb{T}_{33}\boldsymbol{e}_3\otimes\boldsymbol{e}_3\ , \tag{5.2.30}$$

which can be written compactly using the covariant basis as follows (check)

$$\mathbb{T} = \mathbb{T}_{12}(\mathbf{g}_1\otimes\mathbf{g}_2 + \mathbf{g}_2\otimes\mathbf{g}_1) + \mathbb{T}_{33}\boldsymbol{e}_3\otimes\boldsymbol{e}_3 \tag{5.2.31}$$

Finally, employing respectively (5.2.13) and (5.2.14), we obtain

$$\mathbb{T} = \mathbb{T}_{12}(\mathbf{g}_1\otimes\mathbf{g}_2 + \mathbf{g}_2\otimes\mathbf{g}_1) + \mathbb{T}_{33}\boldsymbol{e}_3\otimes\boldsymbol{e}_3 \tag{5.2.32}$$

$$= k(\beta_1\lambda_2^2 - \beta_{-1}\lambda_1^{-2})(\mathbf{g}_1\otimes\mathbf{g}_2 + \mathbf{g}_2\otimes\mathbf{g}_1) + \mathbb{T}_{33}\boldsymbol{e}_3\otimes\boldsymbol{e}_3 \tag{5.2.33}$$

$$= k\left(\frac{\beta_1\lambda_1^2}{1+k^2} - \beta_{-1}\lambda_1^{-2}\right)(\mathbf{g}_1\otimes\mathbf{g}_2 + \mathbf{g}_2\otimes\mathbf{g}_1) + \mathbb{T}_{33}\boldsymbol{e}_3\otimes\boldsymbol{e}_3\ . \tag{5.2.34}$$

This expression holds (assuming the given boundary conditions and deformation) for arbitrary hyperelastic material. We will proceed with the calculation even further for a particular material model, namely the incompressible neo-Hookean model. Before doing so, we need to know how to incorporate incompressibility (or other constraints into our models). We shall do this in the next exercise.

Exercise 13 (Homework). Consider the following block-diagonal form of $\mathbb{B}$, satisfied as we have seen both for the uniaxial tension/compression or for shear (and also simple shear):

$$\mathbb{B} = \begin{pmatrix} \mathbb{B}_{11} & \mathbb{B}_{12} & 0 \\ \mathbb{B}_{12} & \mathbb{B}_{22} & 0 \\ 0 & 0 & \mathbb{B}_{33} \end{pmatrix}\ . \tag{5.2.35}$$

Show that under this assumption on the deformation, general isotropic hyperelastic material satisfies the following "mean-normal stress differences" identity:

$$\frac{\mathbb{T}_{11} - \mathbb{T}_{22}}{\mathbb{T}_{12}} = \frac{\mathbb{B}_{11} - \mathbb{B}_{22}}{\mathbb{B}_{12}}\ . \tag{5.2.36}$$