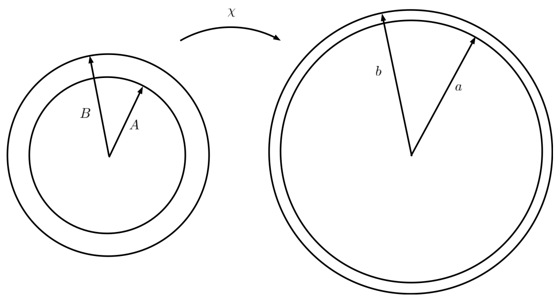
Chapter 7

Inflation of an incompressible hyperelastic balloon



Let us attempt to solve a problem of inflation of a rubber balloon made of a hyperelastic material. We will be interested in characterizing the deformation in terms of the pressure difference outside and inside the ball. We will solve just static problems corresponding to force equilibrium. We will proceed in several steps. The derivation follows closely the one presented in Ogden (1997).

7.1 Spherical deformation of a spherical shell

Let us start by describing the deformation and corresponding deformation gradient. Let us consider deformation as in the sketch, i.e. spherically symmetric inflation/deflation of a spherically symmetric shell of a finite thickness. Due to spherical symmetry, it is natural to work in spherical coordinates. We will distinguish referential (Lagrangian) coordinates R, Θ, Φ and actual (Eulerian) coordinates r, ϑ, φ. The referential shape is given by

$$R \in \langle A, B \rangle, \qquad \Theta \in \langle 0, \pi \rangle, \qquad \Phi \in \langle 0, 2\pi) \,, \tag{7.1.1}$$

and the deformed shape by

$$r \in \langle a, b \rangle, \qquad \vartheta \in \langle 0, \pi \rangle, \qquad \varphi \in \langle 0, 2\pi) \,, \tag{7.1.2}$$

and the deformation $\boldsymbol{\chi}$ in spherical coordinates reads:

$$\boxed{\boldsymbol{\chi}: \qquad r = f(R)R, \qquad \vartheta = \Theta, \qquad \varphi = \Phi} \,, \tag{7.1.3}$$

where $f(R)$ will be determined later.

Let us recall the definition of the deformation gradient in general coordinates. Given Eulerian coordinates ξ^1, ξ^2, ξ^3 and Lagrangian coordinates Ξ^1, Ξ^2, Ξ^3, we can define covariant bases $\{\mathbf{g}_i\}_{i=1}^3$ $\{\mathbf{G}_I\}_{I=1}^3$ as local bases composed of tangent vectors to the coordinate lines:

$$\mathbf{g}_i \stackrel{\text{def}}{=} \frac{\partial \mathbf{x}(\xi^1,\xi^2,\xi^3)}{\partial \xi^i}\,, \qquad i = 1,2,3\,, \tag{7.1.4}$$

$$\mathbf{G}_I \stackrel{\text{def}}{=} \frac{\partial \mathbf{X}(\Xi^1,\Xi^2,\Xi^3)}{\partial \Xi^I}\,, \qquad I = 1,2,3\,, \tag{7.1.5}$$

$$\tag{7.1.6}$$

where $\mathbf{x}(\xi^1,\xi^2,\xi^3)$ and $\mathbf{X}(\Xi^1,\Xi^2,\Xi^3)$ are the position vectors in the actual and reference configuration, respectively. To these covariant bases, we assign their dual (contravariant) counterparts $\{\mathbf{g}^i\}_{i=1}^3$, $\{\mathbf{G}^I\}_{I=1}^3$ by relations

$$\mathbf{g}^i(\mathbf{g}_j) = \delta^i_j\,, \qquad \mathbf{G}^I(\mathbf{G}_J) = \delta^I_J\,, \tag{7.1.7}$$

where δ is the Kronecker symbol. Note that since the three-dimensional space $\mathbb{R}^3$ is naturally endowed with a scalar product, we can represent the dual bases (=forms) as vectors via Riesz' representation theorem and their "action" on vectors is realized via the scalar product, i.e. for instance $\mathbf{g}^i(\mathbf{g}_j) = \mathbf{g}^i \cdot \mathbf{g}_j$, where on the right-hand side, $\mathbf{g}^i$ is understood as the corresponding Riesz representant. So we can think of both the contravariant and covariant bases as vector bases, and the duality relation (7.1.7) expresses mutual orthogonality.

Let us now consider a deformation $\boldsymbol{\chi}$ described by

$$\xi^1 = \chi^1(\Xi^1,\Xi^2,\Xi^3)\,, \tag{7.1.8}$$

$$\xi^2 = \chi^2(\Xi^1,\Xi^2,\Xi^3)\,, \tag{7.1.9}$$

$$\xi^3 = \chi^3(\Xi^1,\Xi^2,\Xi^3)\,, \tag{7.1.10}$$

the corresponding deformation gradient $\mathbb{F}$ is

$$\boxed{\mathbb{F} = \frac{\partial \chi^i}{\partial \Xi^J}\mathbf{g}_i \otimes \mathbf{G}^J}\,. \tag{7.1.11}$$

In our situation, let us take

$$\xi^1 = r\,, \qquad \Xi^1 = R\,, \tag{7.1.12}$$

$$\xi^2 = \vartheta\,, \qquad \Xi^2 = \Theta \tag{7.1.13}$$

$$\xi^3 = \varphi\,, \qquad \Xi^3 = \Phi \tag{7.1.14}$$

and $\boldsymbol{\chi}$ given by (7.1.3). We obtain:

$$\mathbb{F} = (f(R)R)_{,R}\ \mathbf{g}_r \otimes \mathbf{G}^R + \mathbf{g}_\vartheta \otimes \mathbf{G}^\Theta + \mathbf{g}_\varphi \otimes \mathbf{G}^\Phi\,. \tag{7.1.15}$$

Let's express explicitly the bases in spherical coordinates. Since

$$\mathbf{x} = (r\sin\vartheta\cos\varphi, r\sin\vartheta\sin\varphi, r\cos\vartheta) \qquad \text{and} \qquad \mathbf{X} = (R\sin\Theta\cos\Phi, R\sin\Theta\sin\Phi, R\cos\Theta)\,,$$

we get

$$\mathbf{g}_r = \frac{\partial \mathbf{x}}{\partial r} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) \qquad \mathbf{G}_R = \frac{\partial \mathbf{X}}{\partial R} = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta) \tag{7.1.16}$$

$$\mathbf{g}_\vartheta = \frac{\partial \mathbf{x}}{\partial \vartheta} = r(\cos\vartheta\cos\varphi, \cos\vartheta\sin\varphi, -\sin\vartheta) \qquad \mathbf{G}_\Theta = \frac{\partial \mathbf{X}}{\partial \Theta} = R(\cos\Theta\cos\Phi, \cos\Theta\sin\Phi, -\sin\Theta) \tag{7.1.17}$$

$$\mathbf{g}_\varphi = \frac{\partial \mathbf{x}}{\partial \varphi} = r(-\sin\vartheta\sin\varphi, \sin\vartheta\cos\varphi, 0) \qquad \mathbf{G}_\Phi = \frac{\partial \mathbf{X}}{\partial \Phi} = R(-\sin\Theta\sin\Phi, \sin\Theta\cos\Phi, 0) \tag{7.1.18}$$

It is often convenient to use normalized vectors, which we will denote $\mathbf{g}_{\hat{r}}$, $\mathbf{g}_{\hat{\vartheta}}$ $\mathbf{g}_{\hat{\varphi}}$ and $\mathbf{G}_{\hat{R}}$, $\mathbf{G}_{\hat{\Theta}}$ $\mathbf{G}_{\hat{\Phi}}$. Since

$$\|\mathbf{g}_r\| = 1\,, \qquad \|\mathbf{G}_R\| = 1\,, \tag{7.1.19}$$

$$\|\mathbf{g}_\vartheta\| = r\,, \qquad \|\mathbf{G}_\Theta\| = R\,, \tag{7.1.20}$$

$$\|\mathbf{g}_\varphi\| = r\sin\vartheta\,, \qquad \|\mathbf{G}_\Phi\| = R\sin\Theta\,, \tag{7.1.21}$$

and since $\|\mathbf{g}^i\| = \frac{1}{\|\mathbf{g}_i\|}$, $i = 1,2,3$ and $\|\mathbf{G}^I\| = \frac{1}{\|\mathbf{G}_I\|}$, $I = 1,2,3$ due to (7.1.7), we can rewrite $\mathbb{F}$ in the normalized bases as follows

$$\mathbb{F} = (F(R)R)_{,R}\ \mathbf{g}_{\widehat{r}} \otimes \mathbf{G}^{\widehat{R}} + \frac{\|\mathbf{g}_\vartheta\|}{\|\mathbf{G}_\Theta\|}\mathbf{g}_{\widehat{\vartheta}} \otimes \mathbf{G}^{\widehat{\Theta}} + \frac{\|\mathbf{g}_\varphi\|}{\|\mathbf{G}_\Phi\|}\mathbf{g}_{\widehat{\varphi}} \otimes \mathbf{G}^{\widehat{\Phi}} \ . \tag{7.1.22}$$

Plugging-in the deformation (7.1.3), we obtain

$$\mathbb{F} = Rf'(R)\mathbf{g}_{\widehat{r}} \otimes \mathbf{G}^{\widehat{R}} + f(R)(\mathbf{g}_{\widehat{r}} \otimes \mathbf{G}^{\widehat{R}} + \mathbf{g}_{\widehat{\vartheta}} \otimes \mathbf{G}^{\widehat{\Theta}} + \mathbf{g}_{\widehat{\varphi}} \otimes \mathbf{G}^{\widehat{\Phi}}) \ , \tag{7.1.23}$$

note also that for the considered deformation

$$\mathbf{g}_{\widehat{r}} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta) = \mathbf{G}_{\widehat{R}} \ , \tag{7.1.24}$$

$$\mathbf{g}_{\widehat{\vartheta}} = (\cos\vartheta\cos\varphi, \cos\vartheta\sin\varphi, -\sin\vartheta) = (\cos\Theta\cos\Phi, \cos\Theta\sin\Phi, -\sin\Theta) = \mathbf{G}_{\widehat{\Theta}} \ , \tag{7.1.25}$$

$$\mathbf{g}_{\widehat{\varphi}} = (-\sin\varphi, \cos\varphi, 0) = (-\sin\Phi, \cos\Phi, 0) = \mathbf{G}_{\widehat{\Phi}} \ . \tag{7.1.26}$$

Moreover since the bases $\{\mathbf{g}_{\widehat{r}}, \mathbf{g}_{\widehat{\vartheta}}, \mathbf{g}_{\widehat{\varphi}}\}$ and $\{\mathbf{G}_{\widehat{R}}, \mathbf{G}_{\widehat{\Theta}}, \mathbf{G}_{\widehat{\Phi}}\}$ are already ortho-normal, we need not distinguish between covariant and contravariant anymore, to sum up, we have

$$\mathbf{g}_{\widehat{r}} = \mathbf{g}^{\widehat{r}} = \mathbf{G}_{\widehat{R}} = \mathbf{G}^{\widehat{R}} \ , \tag{7.1.27}$$

$$\mathbf{g}_{\widehat{\vartheta}} = \mathbf{g}^{\widehat{\vartheta}} = \mathbf{G}_{\widehat{\Theta}} = \mathbf{G}^{\widehat{\Theta}} \ , \tag{7.1.28}$$

$$\mathbf{g}_{\widehat{\varphi}} = \mathbf{g}^{\widehat{\varphi}} = \mathbf{G}_{\widehat{\Phi}} = \mathbf{G}^{\widehat{\Phi}} \ , \tag{7.1.29}$$

and we can rewrite (7.1.22) in probably the most compact form

$$\boxed{\mathbb{F} = Rf'(R)\mathbf{g}_{\widehat{r}} \otimes \mathbf{G}^{\widehat{R}} + f(R)\mathbb{I}} \ , \tag{7.1.30}$$

where $\mathbb{I}$ is the identity tensor, or in the matrix notation (with respect to any of the normalized bases):

$$\mathbb{F} = \begin{pmatrix} Rf'(R) + f(R) & 0 & 0 \\ 0 & f(R) & 0 \\ 0 & 0 & f(R) \end{pmatrix} . \tag{7.1.31}$$

Since $\mathbb{F}$ is diagonal, it holds

$$\mathbb{F} = \mathbb{U} = \mathbb{V} \ , \tag{7.1.32}$$

and the principal stretches (eigenvalues of $\mathbb{U}$ and $\mathbb{V}$) are

$$\lambda_1 = \lambda_R = \lambda_r = Rf'(R) + f(R) \tag{7.1.33}$$

$$\lambda_2 = \lambda_\vartheta = \lambda_\Theta = f(R) \tag{7.1.34}$$

$$\lambda_3 = \lambda_\varphi = \lambda_\Phi = f(R) \ . \tag{7.1.35}$$

Exercise 17. Find $f(R)$ by assuming *incompressibility*.
Solution: The assumption of incompressibility reads

$$1 = \det\mathbb{F} = \lambda_1\lambda_2\lambda_3 \ , \tag{7.1.36}$$

i.e.

$$(Rf'(R) + f(R))(f(R))^2 = 1 \ , \tag{7.1.37}$$

which can be recast to

$$(Rf(R))^2(Rf(R))' = R^2 \ . \tag{7.1.38}$$

This relation can be easily integrated between the inner radius A and $R \in \langle A, B\rangle$:

$$R^3 f(R)^3 - \underbrace{A^3 f(A)^3}_{=a^3} = R^3 - A^3 \,, \tag{7.1.39}$$

which yields

$$\boxed{f(R) = \left(1 + \frac{a^3 - A^3}{R^3}\right)^{\frac{1}{3}}} \,. \tag{7.1.40}$$

So we have an explicit description of the deformation and of the principal stretches. Let us define

$$\lambda \overset{\text{def}}{=} \lambda_2 = \lambda_3 = \lambda_1^{-\frac{1}{2}} \,. \tag{7.1.41}$$

and also

$$\lambda_a \overset{\text{def}}{=} \lambda(R{=}A) = \frac{a}{A} \,, \qquad \lambda_b \overset{\text{def}}{=} \lambda(R{=}B) = \frac{b}{B} \,, \tag{7.1.42}$$

Then (7.1.40) implies

$$\lambda_a^3 - 1 = (\lambda^3(R) - 1)\left(\frac{R}{A}\right)^3 = (\lambda_b^3 - 1)\left(\frac{B}{A}\right)^3 \,. \tag{7.1.43}$$

Exercise 18. Show that the above identity implies that either $\lambda_a \geq \lambda \geq \lambda_b \geq 1$, or $\lambda_a \leq \lambda \leq \lambda_b \leq 1$, the first case corresponding to inflation, the second to deflation.

7.2 Stress in the shell

The spherical symmetry of the deformation implies also spherical symmetry of the stress field. This implies that the first Piola-Kirchhoff stress $\mathbb{T}^{(1)}$ and the Cauchy stress $\mathbb{T}$ must be of the following form:

$$\mathbb{T}^{(1)} = \sigma_1^{(1)}(R)\, \mathbf{g}_{\hat{r}} \otimes \mathbf{G}^{\hat{R}} + \sigma_2^{(1)}(R)\, (\mathbb{I} - \mathbf{g}_{\hat{r}} \otimes \mathbf{G}^{\hat{R}}) \,, \tag{7.2.1}$$

$$\mathbb{T} = \sigma_1(R)\, \mathbf{g}_{\hat{r}} \otimes \mathbf{g}^{\hat{r}} + \sigma_2(R)\, (\mathbb{I} - \mathbf{g}_{\hat{r}} \otimes \mathbf{g}^{\hat{r}}) \,, \tag{7.2.2}$$

with $\sigma_1^{(1)}$, $\sigma_2^{(1)}(=\sigma_3^{(1)})$ and σ_1, $\sigma_2(=\sigma_3)$ the corresponding principal stresses. Since $\mathbb{T}^{(1)} = J\mathbb{T}\mathbb{F}^{-T}$ and $J = 1$ (incompressibility) and

$$\mathbb{F}^{-T} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}^{-T} = \begin{pmatrix} \lambda^{-2} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}^{-T} = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}, \tag{7.2.3}$$

it holds

$$\sigma_1^{(1)} = \sigma_1 \lambda^2 \,, \qquad \sigma_2^{(1)} = \sigma_2 \lambda^{-1} \,. \tag{7.2.4}$$

Note that since $\mathbb{F}^{-T}$ is diagonal, the first Piola Kirchhoff stress is symmetric (which as we know, it is not the case in general).

The balance of linear momentum corresponding to the static equilibrium in the absence of body forces reads

$$\operatorname{Div} \mathbb{T}^{(1)} = \mathbf{0} \,, \qquad \text{or} \qquad \operatorname{div} \mathbb{T} = \mathbf{0} \tag{7.2.5}$$

In spherical coordinates, the divergence of a symmetric tensor $\mathbb{A}$ reads as follows:

$$\begin{aligned}(\operatorname{div} \mathbb{A}) = &\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \mathbb{A}_{\hat{r}\hat{r}}) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta \mathbb{A}_{\hat{\vartheta}\hat{r}}) + \frac{1}{r\sin\vartheta}\frac{\partial \mathbb{A}_{\hat{\varphi}\hat{r}}}{\partial\varphi} - \frac{1}{r}(\mathbb{A}_{\hat{\vartheta}\hat{\vartheta}} - \mathbb{A}_{\hat{\varphi}\hat{\varphi}})\right)\mathbf{g}_{\hat{r}} \\ &+ \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \mathbb{A}_{\hat{r}\hat{\vartheta}}) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta \mathbb{A}_{\hat{\vartheta}\hat{\vartheta}}) + \frac{1}{r\sin\vartheta}\frac{\partial \mathbb{A}_{\hat{\varphi}\hat{\vartheta}}}{\partial\varphi} - \frac{1}{r}(\mathbb{A}_{\hat{\vartheta}\hat{r}} - \cot\vartheta \mathbb{A}_{\hat{\varphi}\hat{\varphi}})\right)\mathbf{g}_{\hat{\vartheta}} \\ &+ \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \mathbb{A}_{\hat{r}\hat{\varphi}}) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta \mathbb{A}_{\hat{\vartheta}\hat{\varphi}}) + \frac{1}{r\sin\vartheta}\frac{\partial \mathbb{A}_{\hat{\varphi}\hat{\varphi}}}{\partial\varphi} - \frac{1}{r}(\mathbb{A}_{\hat{\varphi}\hat{r}} + \cot\vartheta \mathbb{A}_{\hat{\varphi}\hat{\vartheta}})\right)\mathbf{g}_{\hat{\varphi}} \,,\end{aligned} \tag{7.2.6}$$

and similarly for the Lagrangian version.

Exercise 19. Express the equilibrium equations (7.2.5) for the stress field given by (7.2.1). Solution: Due to the simple structure of $\mathbb{T}^{(1)}$ and $\mathbb{T}$ in our application, the equilibrium conditions reduce to:

$$\boxed{\frac{d\sigma_1^{(1)}}{dR} + \frac{2}{R}(\sigma_1^{(1)} - \sigma_2^{(1)}) = 0}\ , \tag{7.2.7}$$

or, equivalently

$$\frac{d\sigma_1}{dr} + \frac{2}{r}(\sigma_1 - \sigma_2) = 0\ . \tag{7.2.8}$$

For an incompressible hyperelastic material characterized by strain energy function W, we have shown that

$$\mathbb{T}^{(1)} = -Jp\mathbb{F}^{-T} + 2\frac{\partial W}{\partial \mathbb{B}}\mathbb{F}\ , \tag{7.2.9}$$

where p is the "constraint pressure". We also learned to express the energy-related part in terms of principal strains, i.e. having W given by $W = \widehat{W}(\lambda_1, \lambda_2, \lambda_3)$, we can express

$$2\frac{\partial W}{\partial \mathbb{B}}\mathbb{F} = 2\sum_{\alpha=1}^{3} \frac{\partial \hat{W}}{\partial \lambda_\alpha}\frac{\partial \lambda_\alpha}{\partial \mathbb{B}}\mathbb{F} = 2\sum_{\alpha=1}^{3} \frac{\partial \hat{W}}{\partial \lambda_\alpha}\frac{1}{2\lambda_\alpha}(\mathbf{g}_\alpha^{\mathbb{B}} \otimes \mathbf{g}_\alpha^{\mathbb{B}})\mathbb{F}\ , \tag{7.2.10}$$

where $\mathbf{g}_\alpha^{\mathbb{B}}$ are unit eigenvectors of $\mathbb{B}$ (forming orthonormal local basis). Since for our deformation

$$\mathbb{F} = \mathbb{V} = \sum_{\alpha=1}^{3} \lambda_\alpha \mathbf{g}_\alpha^{\mathbb{B}} \otimes \mathbf{g}_\alpha^{\mathbb{B}} = \lambda_r \mathbf{g}_{\hat{r}} \otimes \mathbf{g}_{\hat{r}} + \lambda_\vartheta \mathbf{g}_{\hat{\vartheta}} \otimes \mathbf{g}_{\hat{\vartheta}} + \lambda_\varphi \mathbf{g}_{\hat{\varphi}} \otimes \mathbf{g}_{\hat{\varphi}}\ , \tag{7.2.11}$$

we get

$$2\frac{\partial W}{\partial \mathbb{B}}\mathbb{F} = \sum_{\alpha=1}^{3} \frac{\partial \hat{W}}{\partial \lambda_\alpha}\mathbf{g}_\alpha^{\mathbb{B}} \otimes \mathbf{g}_\alpha^{\mathbb{B}} = \frac{\partial \hat{W}}{\partial \lambda_r}\mathbf{g}_{\hat{r}} \otimes \mathbf{g}_{\hat{r}} + \frac{\partial \hat{W}}{\partial \lambda_\vartheta}\mathbf{g}_{\hat{\vartheta}} \otimes \mathbf{g}_{\hat{\vartheta}} + \frac{\partial \hat{W}}{\partial \lambda_\varphi}\mathbf{g}_{\hat{\varphi}} \otimes \mathbf{g}_{\hat{\varphi}}\ . \tag{7.2.12}$$

Finally, since $J = 1$ and $\mathbb{F}^{-T}$ is given by (7.2.3), we obtain

$$\sigma_1^{(1)} = \mathbb{T}^{(1)}_{\hat{r}\hat{R}} = -p\lambda_1^{-1} + \frac{\partial \hat{W}}{\partial \lambda_1}\ , \tag{7.2.13}$$

$$\sigma_2^{(1)} = \mathbb{T}^{(1)}_{\hat{\vartheta}\hat{\Theta}} = \mathbb{T}^{(1)}_{\hat{\varphi}\hat{\Phi}} = -p\lambda_2^{-1} + \frac{\partial \hat{W}}{\partial \lambda_2}\ , \tag{7.2.14}$$

both evaluated for $\lambda_1 = \lambda^{-2}$, $\lambda_2 = \lambda_3 = \lambda$.

Exercise 20. Both principal stresses are now functions of λ and p, let us transform the equilibrium condition in terms of λ. Solution: Since we have explicit relation

$$\lambda = f(R) = \left(1 + \frac{a^3 - A^3}{R^3}\right)^{\frac{1}{3}}\ , \tag{7.2.15}$$

we get

$$\frac{d\lambda}{dR} = \frac{1}{3}\left(1 + \frac{a^3 - A^3}{R^3}\right)^{-\frac{2}{3}}\left(\frac{a^3 - A^3}{R^4}\right)(-3)\ , \tag{7.2.16}$$

which implies that

$$R\frac{d\lambda}{dR} = -\frac{1}{\lambda^2}\underbrace{\frac{a^3 - A^3}{R^3}}_{=\lambda^3 - 1} = \lambda^{-2} - \lambda\ . \tag{7.2.17}$$

Let us rewrite equilibrium condition (7.2.7) first equivalently as

$$\frac{d\sigma_1^{(1)}}{d\lambda}\frac{d\lambda}{dR} = -\frac{2}{R}(\sigma_1^{(1)} - \sigma_2^{(1)}) , \tag{7.2.18}$$

multiply by R and employ (7.2.17). We arrive at

$$\boxed{\frac{d\sigma_1^{(1)}}{d\lambda} = 2\frac{\sigma_1^{(1)} - \sigma_2^{(1)}}{\lambda - \lambda^{-2}}} \tag{7.2.19}$$

Boundary condition in our problem are stress-free upper boundary and internal pressure $-P$ (which we actually seek as a function of the deformation). This is best expressed in terms of the Cauchy principal stress σ_1:

$$\sigma_1(r = b) = 0 , \tag{7.2.20a}$$

$$\sigma_1(r = a) = -P . \tag{7.2.20b}$$

Note that we could also have external pressure p_0 and the sought quantity would be just the overpressure (difference between the internal and external pressure), but let us take $p_0 = 0$, for simplicity. In view of (7.2.4), in terms of principal stresses $\sigma_1^{(1)}$, this equivalently reads

$$\sigma_1^{(1)}(R = B) = \sigma_1^{(1)}(\lambda_b) = 0 , \tag{7.2.21a}$$

$$\sigma_1^{(1)}(R = A) = \sigma_1^{(1)}(\lambda_a) = -P\lambda_a^2 . \tag{7.2.21b}$$

Let us now define

$$\widetilde{W}(\lambda) \stackrel{\text{def}}{=} \hat{W}(\lambda^{-2}, \lambda, \lambda) . \tag{7.2.22}$$

Exercise 21. Show that

$$\boxed{\widetilde{W}'(\lambda) = 2(\sigma_2^{(1)} - \lambda^{-3}\sigma_1^{(1)})}. \tag{7.2.23}$$

Solution:

$$\widetilde{W}'(\lambda) = \frac{\partial\hat{W}}{\partial\lambda_1}\left(\frac{-2}{\lambda^3}\right) + 2\frac{\partial\hat{W}}{\partial\lambda_2} = 2\left(-\lambda^{-3}\left(\sigma_1^{(1)} + p\lambda^2\right) + \left(\sigma_2^{(1)} + p\lambda^{-1}\right)\right) = 2(\sigma_2^{(1)} - \lambda^{-3}\sigma_1^{(1)}) . \tag{7.2.24}$$

Let us now employ this relation in the equilibrium equation (7.2.19):

$$\frac{d\sigma_1^{(1)}}{d\lambda} = 2\frac{\sigma_1^{(1)} - \sigma_2^{(1)}}{\lambda - \lambda^{-2}} = 2\left(\frac{\lambda^{-3}\sigma_1^{(1)} - \sigma_2^{(1)}}{\lambda - \lambda^{-2}} + \sigma_1^{(1)}\frac{1 - \lambda^{-3}}{\lambda - \lambda^{-2}}\right) = -\frac{\widetilde{W}'(\lambda)}{\lambda - \lambda^{-2}} + 2\frac{\sigma_1^{(1)}}{\lambda} . \tag{7.2.25}$$

Let us put the last term on the left hand side

$$\frac{d\sigma_1^{(1)}}{d\lambda} - 2\frac{d\sigma_1^{(1)}}{\lambda} = \lambda^2\frac{d}{d\lambda}\left(\frac{\sigma_1^{(1)}}{\lambda^2}\right) , \tag{7.2.26}$$

which finally yields

$$\frac{d}{d\lambda}\left(\frac{\sigma_1^{(1)}}{\lambda^2}\right) = -\frac{\widetilde{W}'(\lambda)}{\lambda^3 - 1} . \tag{7.2.27}$$

Integrating across the shell, i.e. from λ_a to λ_b, gives

$$\underbrace{\frac{\sigma^{(1)}(\lambda_b)}{\lambda_b^2}}_{=0} - \underbrace{\frac{\sigma^{(1)}(\lambda_a)}{\lambda_a^2}}_{=-P} = \int_{\lambda_b}^{\lambda_a} \frac{\widetilde{W}'(\lambda)}{\lambda^3 - 1} d\lambda , \tag{7.2.28}$$

where we employed the boundary conditions (7.2.21). So finally, we have expression for the overpressure P depending only on the energy function $\widetilde{W}$:

$$\boxed{P = \int_{\lambda_b}^{\lambda_a} \frac{\widetilde{W}'(\lambda)}{\lambda^3 - 1} d\lambda} \tag{7.2.29}$$

This expression is exact and holds for incompressible hyperelastic spherical shell subjected to spherical deformation. Provided that we pick some particular material by specifying suitable $W(\lambda_1, \lambda_2, \lambda_3)$, after identifying $\widetilde{W}(\lambda)$ and after carrying out the integration, we would find out the equilibrium overpressure in the shell corresponding to the given deformation.

Let us now push the calculation a bit further for a simplified setting, corresponding to the so-called thin shell.

7.3 Thin shell solution

Let us define

$$\frac{B-A}{A} = \frac{\varepsilon}{3} \qquad \varepsilon \ll 1 \, . \tag{7.3.1}$$

Rewriting the equilibrium equation (7.2.7) as

$$\frac{d}{dR}\left(R^2 \sigma_1^{(1)}\right) = 2R\sigma_2^{(1)} \tag{7.3.2}$$

and integrating between $R = A$ and $R = B$ yields

$$\underbrace{B^2\sigma_1^{(1)}(R=B)}_{=0 \;\; (BC)} - A^2\sigma_1^{(1)}(R=A) = 2\int_A^B R\sigma_2^{(1)}(R)dR = \sigma_2^{(1)}(R^*)(B^2 - A^2) \, , \tag{7.3.3}$$

where we employed the integral mean-value theorem, and $R^* \in (A,B)$ is the corresponding internal point. So we obtained

$$-\sigma_1^{(1)}(A) = \sigma_2^{(1)}(R^*)\frac{B^2 - A^2}{A^2} = \sigma_2^{(1)}(R^*)\underbrace{\frac{B-A}{A}}_{\varepsilon/3}\underbrace{\frac{B+A}{A}}_{\doteq 2} \doteq \frac{2}{3}\varepsilon\sigma_2^{(1)}(R^*) \, . \tag{7.3.4}$$

In the so-called membrane approximation, we shall assume that $\sigma_2^{(1)}$ is constant across the thin shell, so we get

$$\boxed{-\sigma_1^{(1)} \doteq \frac{2}{3}\varepsilon\sigma_2^{(1)}} \, . \tag{7.3.5}$$

Let us now employ the mean value theorem also to (7.2.29):

$$P = \int_{\lambda_b}^{\lambda_a} \frac{\widetilde{W}'(\lambda)}{\lambda^3 - 1} d\lambda = \frac{\widetilde{W}'(\lambda^*)}{(\lambda^*)^3 - 1}(\lambda_a - \lambda_b) \, . \tag{7.3.6}$$

Recalling the identities (7.1.43), we can express

$$\lambda_a = \left(1 + (\lambda_b^3 - 1)\left(1 + \underbrace{\frac{B-A}{A}}_{=\varepsilon/3}\right)^3\right)^{\frac{1}{3}} \doteq \left(1 + (\lambda_b^3 - 1)(1+\varepsilon)\right)^{\frac{1}{3}} = \lambda_b(1 + \varepsilon(1 - \lambda_b^{-3}))^{\frac{1}{3}} \doteq \lambda_b\left(1 + \frac{\varepsilon}{3}(1 - \lambda_b^{-3})\right) \, , \tag{7.3.7}$$

which yields

$$\boxed{\lambda_a - \lambda_b \doteq \frac{\varepsilon}{3}(\lambda_b - \lambda_b^{-2})} . \tag{7.3.8}$$

Using this expression in (7.3.6) gives

$$P \doteq \frac{\widetilde{W}'(\lambda^*)}{(\lambda^*)^3-1}\,\frac{\varepsilon}{3}(\lambda_b-\lambda_b^{-2})\,. \tag{7.3.9}$$

Since in the thin-shell scenario $\lambda_a \doteq \lambda_b$ and λ^* is between the two values, we don not need to distinguish between them and we arrive at the following thin-shell approximation of the overpressure

$$\boxed{P \doteq \varepsilon\frac{\widetilde{W}'(\lambda)}{3\lambda^2}}\,. \tag{7.3.10}$$

Using relation (7.2.23) and the above expression together with (7.3.5), we get

$$\sigma_2^{(1)}(\lambda) = \frac{\widetilde{W}'(\lambda)}{2} + \lambda^{-3}\sigma_1^{(1)}(\lambda) \Longrightarrow \sigma_2^{(1)}(\lambda) = \frac{\widetilde{W}'(\lambda)}{2}(1+O(\varepsilon))\,, \tag{7.3.11}$$

so we take

$$\boxed{\sigma_2^{(1)} \doteq \frac{\widetilde{W}'(\lambda)}{2}}\,. \tag{7.3.12}$$

Let us finally define the surface tension T (dimension force per length) as the total force per line segment cutting through the shell:

$$T \stackrel{\text{def}}{=} (b-a)\sigma_2 = \underbrace{\frac{b-a}{B-A}}_{=\lambda_1}(B-A)\underbrace{\sigma_2}_{=\sigma_2^{(1)}\lambda_2} = (B-A)\lambda_1\lambda_2\sigma_2^{(1)}\,, \tag{7.3.13}$$

which gives

$$\boxed{T = \frac{B-A}{2}\frac{\widetilde{W}'(\lambda)}{\lambda} = \frac{A\varepsilon}{6}\frac{\widetilde{W}'(\lambda)}{\lambda}}\,. \tag{7.3.14}$$

Note that this implies

$$\boxed{P = \frac{2T}{A\lambda} = \frac{2T}{a}}\,, \tag{7.3.15}$$

which is the standard Young-Laplace formula for overpressure inside a membrane of radius a with surface tension T.

7.4 Rubber balloon

Let us finish by specifying the strain energy W for a rubber-like material.

Exercise 22. Identify the overpressure and surface tension for Ogden material. Solution: For Ogden material, the strain energy function (in terms of principal stretches) as follows

$$\hat{W}(\lambda_1,\lambda_2,\lambda_3) = \sum_{k=1}^{N}\frac{\mu_k}{\alpha_k}(\lambda_1^{\alpha_k}+\lambda_2^{\alpha_k}+\lambda_3^{\alpha_k}-3)\,, \tag{7.4.1}$$

which gives

$$\widetilde{W}(\lambda) = \hat{W}(\lambda^{-2},\lambda,\lambda) = \sum_{k=1}^{N}\frac{\mu_k}{\alpha_k}(\lambda^{-2\alpha_k}+2\lambda^{\alpha_k}-3)\,, \tag{7.4.2}$$

so that

$$\widetilde{W}'(\lambda) = \sum_{k=1}^{N}\mu_k(-2\lambda^{-2\alpha_k-1}+2\lambda^{\alpha_k-1})\,, \tag{7.4.3}$$

which gives

$$\boxed{P \doteq \frac{\varepsilon}{3}\frac{\widetilde{W}'(\lambda)}{\lambda^2} = \frac{2\varepsilon}{3}\sum_{k=1}^{N}\mu_k(\lambda^{\alpha_k-3} - \lambda^{-2\alpha_k-3})}\,, \tag{7.4.4}$$

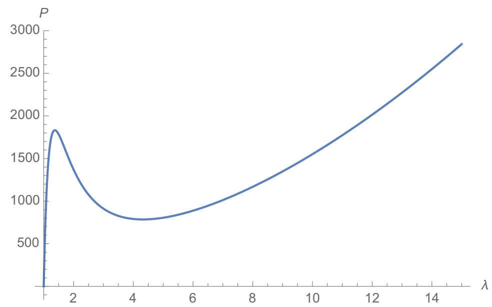
and

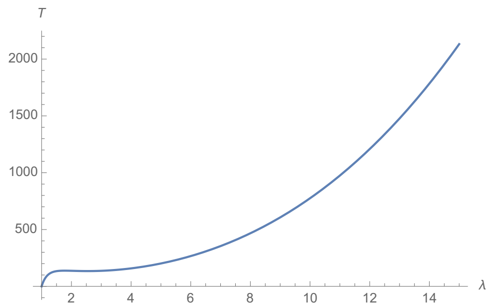
$$\boxed{T = \frac{A\lambda}{2}P = \frac{A\varepsilon}{3}\sum_{k=1}^{N}\mu_k(\lambda^{\alpha_k-2} - \lambda^{-2\alpha_k-2})}\,, \tag{7.4.5}$$

For an Ogden rubber-like material with $N = 3$, specified in Holzapfel (2000) as follows: and taking $\varepsilon =$

$$\begin{array}{ll} \alpha_1 = 1.3 & \mu_1 = 6.3 \times 10^5\ Nm^{-2} \\ \alpha_2 = 5.0 & \mu_2 = 0.012 \times 10^5\ Nm^{-2} \\ \alpha_3 = -2.0 & \mu_3 = -0.1 \times 10^5\ Nm^{-2} \end{array}$$

0.01, $A = 0.1m$ the response looks like this:





The fact that the dependence of overpressure P on the stretch λ is non-monotone is characteristic for rubbers. In this example, if one increases the pressure when inflating the balloon, the resistance largest at the beginning - this corresponds to the initial steep monotone part of the graph. Then at some point ($\lambda \sim 1$), the stretch rapidly increases jumping over the non-monotone and non-stable part of the response to a state witch significantly larger stretch ($\lambda \sim 10$), and finally, with with further increasing the overpressure, the stretch again increases monotonously. While the numbers need not be representative, this actual behavior of rubber balloons is observed in reality.