5. MOVING SPATIAL FRAME

5.1 Observer transformation

In Chapter 6, we will require that the form of the constitutive equations is independent of the movement of an observer. The notion of frame helps us to formulate this requirement mathematically. A *frame* can be understood as an observer who is equipped to measure position in Euclidean space. To every frame belongs a reference point, the so-called *origin*, from which an observer measures distances or defines position vectors in space.

Let us consider two different frames, one fixed (unstarred) and the other one in motion (starred). Both are later considered to describe the present configuration of a material body and are, therefore, called the spatial frames. Figure 5.1 shows two such frames and the relationship between the position vectors of the same observer measured in both frame. Let $\vec{x}$ be a position vector of the observer P in the present configuration relative to the fixed frame and $\vec{x}^{**}$ the position vector of the same observer in the moving frame. They are hence connected by the relationship

$$\vec{x}^{**} = \vec{x} + \vec{b}^{**} , \tag{5.1}$$

where $\vec{b}^{**}$ gives the displacement between the unstarred and the starred reference points (origins). Notice that this relation is independent of the choice of *coordinate system*. However, the observer may refer his position vector not only to the origin, but also to a coordinate system, which is attached to the origin. The coordinate system can be chosen arbitrarily or matched to a realistic situation. In Figure 5.1, these coordinate systems are Cartesian.

Present configuration of $\mathcal{B}$

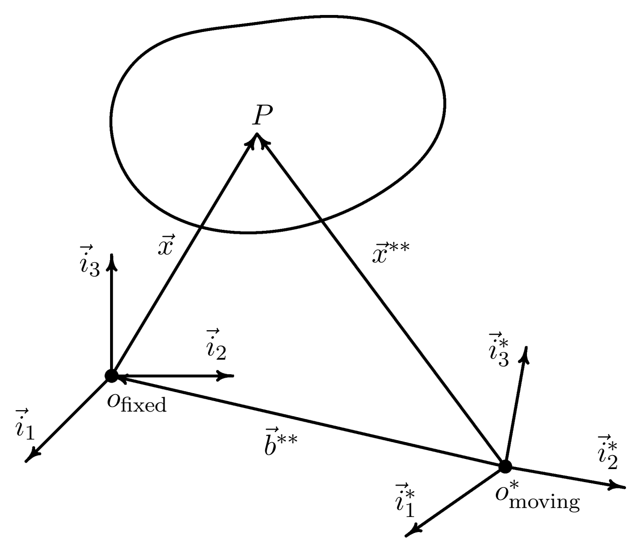

Figure 5.1. Observer transformation.

The component form of (5.1) is

$$x^*_{k^*}\vec{i}^*_{k^*} = x_k\vec{i}_k + b^*_{k^*}\vec{i}^*_{k^*} . \tag{5.2}$$

The Cartesian base vectors $\vec{i}_k$ and $\vec{i}^*_{k^*}$ associated with the unstarred and starred spatial frames are related by (1.12):

$$\vec{i}_k = \delta_{kk^*}\vec{i}^*_{k^*} , \quad \vec{i}^*_{k^*} = \delta_{k^*k}\vec{i}_k , \tag{5.3}$$

where δ_{kk^*} and δ_{k^*k} are the shifters between the two spatial frames. Substituting for $\vec{i}_k$ from $(5.3)_1$ into (5.2) and comparing the components at $\vec{i}^*_{k^*}$, we obtain

$$x^*_{k^*} = \delta_{kk^*}x_k + b^*_{k^*} . \tag{5.4}$$

Multiplying by $\vec{i}_{k^*}$ and defining two vectors

$$\vec{x}^* := x^*_{k^*}\vec{i}_{k^*} , \qquad \vec{b}^* := b^*_{k^*}\vec{i}_{k^*} , \tag{5.5}$$

we obtain [1]

$$\vec{x}^* = \delta_{kk^*} x_k \vec{i}_{k^*} + \vec{b}^* \; . \tag{5.6}$$

Making use of $x_k = \vec{i}_k \cdot \vec{x}$, the identity $\vec{i}_{k^*}(\vec{i}_k \cdot \vec{x}) = (\vec{i}_{k^*} \otimes \vec{i}_k) \cdot \vec{x}$ and introducing tensor $\boldsymbol{O}$,

$$\boldsymbol{O} := O_{k^*k}(\vec{i}_{k^*} \otimes \vec{i}_k) \; , \qquad O_{k^*k} = \delta_{kk^*} \; , \tag{5.7}$$

equation (5.6) may be rewritten in invariant notation as

$$\vec{x}^* = \boldsymbol{O}(t) \cdot \vec{x} + \vec{b}^*(t) \; . \tag{5.8}$$

This equation shows that the same point can be represented by its components x_k^* in the moving coordinate system as well as by x_k in the fixed coordinate system. This expresses a rigid motion of the starred spatial frame. In fact, $\vec{b}^*(t)$ corresponds to the translation and $\boldsymbol{O}(t)$ to the rotation of this frame. As indicated, both $\vec{b}^*(t)$ and $\boldsymbol{O}(t)$ can be time-dependent. The transformation (5.8) is often referred to as the *observer transformation* or the *Euclidean transformation* $(x,t) \rightarrow (x^*,t)$. [2]

In view of $(5.3)_2$, the tensor $\boldsymbol{O}(t)$ may be expressed as the tensor product of the starred and unstarred base vectors:

$$\boldsymbol{O}(t) = \vec{i}_k^*(t) \otimes \vec{i}_k \; . \tag{5.9}$$

The transposed tensor to $\boldsymbol{O}(t)$ is

$$\boldsymbol{O}^T(t) = \delta_{k^*k}(\vec{i}_{k^*} \otimes \vec{i}_k) = \vec{i}_k \otimes \vec{i}_k^*(t) \; . \tag{5.10}$$

This implies that $\boldsymbol{O}(t)$ is an orthogonal tensor since

$$\boldsymbol{O}(t) \cdot \boldsymbol{O}^T(t) = \boldsymbol{O}^T(t) \cdot \boldsymbol{O}(t) = \boldsymbol{I} \; , \tag{5.11}$$

where $\boldsymbol{I}$ is the identity tensor. Strictly speaking, there are two identity tensors, one in the unstarred frame, $\boldsymbol{I} = \vec{i}_k \otimes \vec{i}_k$, and one in the starred frame $\boldsymbol{I}^* = \vec{i}_k^* \otimes \vec{i}_k^*$; we shall, however, disregard this subtlety.

We say that a scalar-, vector- and tensor-valued quantity ϕ is *objective* or *frame indifferent* if it is invariant under all observer transformation (5.8), that is, if $\phi^{**} = \phi$. For instance, tensor $\boldsymbol{a}$ is objective if its components transform under the observer transformation (5.8) according to the relation

$$a^*_{k^*l^*} = O_{k^*k}(t) a_{kl} O_{l^*l}(t) \; , \tag{5.12}$$

where a_{kl} and $a^*_{k^*l^*}$ are components of $\boldsymbol{a}$ relative to the unstarred and starred frames, respectively. To see it, let us rewrite the transformation relation (5.3) for the base vectors in terms of the components of the tensor $\boldsymbol{O}$. By $(5.7)_2$, the relation (5.3) may be rewritten in the form

$$\vec{i}_k = O_{k^*k} \vec{i}^*_{k^*} \; , \qquad \vec{i}^*_{k^*} = O_{kk^*} \vec{i}_k \; . \tag{5.13}$$

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[1] Note that we distinguish between three different vectors, $\vec{x} = x_k \vec{i}_k$, $\vec{x}^* = x_k^* \vec{i}_k$ and $\vec{x}^{**} = x_k^* \vec{i}_k^*$. Vector notation becomes ambiguous if the vectors $\vec{x}^*$ and $\vec{x}^{**}$ are denoted by the same symbol $\vec{x}^*$. Compare (5.1) and (5.8) in this case.

[2] The most general change of frame $(x,t) \rightarrow (x^*,t^*)$ is, in addition, characterized by a shift in time:

$$t^* = t - a \; ,$$

where a is a particular time.

Then

$$\boldsymbol{a} = a_{kl}(\vec{i}_k \otimes \vec{i}_l) = a_{kl}O_{k^*k}O_{l^*l}(\vec{i}^*_{k^*} \otimes \vec{i}^*_{l^*}) \stackrel{!}{=} \boldsymbol{a}^{**} = a^*_{k^*l^*}(\vec{i}^*_{k^*} \otimes \vec{i}^*_{l^*}) \ ,$$

which yields (5.12). Introducing tensor $\boldsymbol{a}^*$,

$$\boldsymbol{a}^* := a^*_{k^*l^*}(\vec{i}_{k^*} \otimes \vec{i}_{l^*}) \ , \tag{5.14}$$

the component form (5.12) may be written in invariant form [3]

$$\boldsymbol{a}^* = \boldsymbol{O}(t) \cdot \boldsymbol{a} \cdot \boldsymbol{O}^T(t) \ . \tag{5.15}$$

In an analogous way, a scalar- and vector-valued physical quantities λ and $\vec{u}$ are called objective if they transform under a rigid motion of spatial frame according to

$$\begin{aligned} \lambda^* &= \lambda \ , \\ u^*_{k^*} &= O_{k^*k}(t)u_k \ , \qquad \text{or, invariantly,} \qquad \vec{u}^{\,*} = \boldsymbol{O}(t) \cdot \vec{u} \ . \end{aligned} \tag{5.16}$$

5.2 Objectivity of some geometric objects

Let us now examine the objectivity property of different geometric objects. We begin with the Eulerian velocity $\vec{v}$ and the Eulerian acceleration $\vec{a}$. Suppose that the motion is represented in the unstarred frame by (1.29). Then, in view of (5.8), it is given in the starred frame by

$$\vec{\chi}^*(\vec{X}, t) = \boldsymbol{O}(t) \cdot \vec{\chi}(\vec{X}, t) + \vec{b}^*(t) \ . \tag{5.17}$$

where $\vec{\chi}^*(\vec{X}, t) := \chi^*_{k^*}(\vec{X}, t)\,\vec{i}_{k^*}$. Differentiation of (5.17) with respect to t yields the following connection between the velocities and accelerations in the starred and unstarred frames:

$$\vec{v}^*(\vec{x}^*, t) = \boldsymbol{O}(t) \cdot \vec{v}(\vec{x}, t) + \dot{\boldsymbol{O}}(t) \cdot \vec{x} + \dot{\vec{b}}^*(t) \ , \tag{5.18}$$

$$\vec{a}^*(\vec{x}^*, t) = \boldsymbol{O}(t) \cdot \vec{a}(\vec{x}, t) + 2\,\dot{\boldsymbol{O}}(t) \cdot \vec{v}(\vec{x}, t) + \ddot{\boldsymbol{O}}(t) \cdot \vec{x} + \ddot{\vec{b}}^*(t) \ . \tag{5.19}$$

Let us introduce the angular velocity tensor $\boldsymbol{\Omega}$ which represents the spin of the starred frame with respect to the unstarred frame:

$$\boldsymbol{\Omega}(t) := \dot{\boldsymbol{O}}(t) \cdot \boldsymbol{O}^T(t) \ . \tag{5.20}$$

The relation

$$\boldsymbol{0} = (\boldsymbol{O} \cdot \boldsymbol{O}^T)^{\bullet} = \dot{\boldsymbol{O}} \cdot \boldsymbol{O}^T + \boldsymbol{O} \cdot \dot{\boldsymbol{O}}^T = \dot{\boldsymbol{O}} \cdot \boldsymbol{O}^T + (\dot{\boldsymbol{O}} \cdot \boldsymbol{O}^T)^T = \boldsymbol{\Omega} + \boldsymbol{\Omega}^T$$

shows that $\boldsymbol{\Omega}$ is a skew-symmetric tensor. Moreover,

$$\dot{\boldsymbol{\Omega}} = (\dot{\boldsymbol{O}} \cdot \boldsymbol{O}^T)^{\bullet} = \ddot{\boldsymbol{O}} \cdot \boldsymbol{O}^T + \dot{\boldsymbol{O}} \cdot \dot{\boldsymbol{O}}^T = \ddot{\boldsymbol{O}} \cdot \boldsymbol{O}^T + \dot{\boldsymbol{O}} \cdot (\boldsymbol{O}^T \cdot \boldsymbol{O}) \cdot \dot{\boldsymbol{O}}^T = \ddot{\boldsymbol{O}} \cdot \boldsymbol{O}^T + \boldsymbol{\Omega} \cdot \boldsymbol{\Omega}^T = \ddot{\boldsymbol{O}} \cdot \boldsymbol{O}^T - \boldsymbol{\Omega} \cdot \boldsymbol{\Omega} \ ,$$

which yields

$$\ddot{\boldsymbol{O}} \cdot \boldsymbol{O}^T = \dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \cdot \boldsymbol{\Omega} \ . \tag{5.21}$$

[3] Note again that we distinguish between three different tensors $\boldsymbol{a}$, $\boldsymbol{a}^*$ and $\boldsymbol{a}^{**}$.

With the aid of (5.8), (5.20) and (5.21), the transformation formulae for the velocity and acceleration can be expressed in the forms

$$\vec{v}^* = \boldsymbol{O} \cdot \vec{v} + \boldsymbol{\Omega} \cdot (\vec{x}^* - \vec{b}^*) + \dot{\vec{b}}^* , \tag{5.22}$$

$$\vec{a}^* = \boldsymbol{O} \cdot \vec{a} + 2\boldsymbol{\Omega} \cdot (\vec{v}^* - \dot{\vec{b}}^*) - \boldsymbol{\Omega} \cdot \boldsymbol{\Omega} \cdot (\vec{x}^* - \vec{b}^*) + \dot{\boldsymbol{\Omega}} \cdot (\vec{x}^* - \vec{b}^*) + \ddot{\vec{b}}^* . \tag{5.23}$$

Inspection of these equations shows that both the velocity and the acceleration are not objective vectors. The additional terms causing the failure of objectivity have the following names:

$\boldsymbol{\Omega} \cdot (\vec{x}^* - \vec{b}^*)$ - relative angular velocity of the starred frame with respect to unstarred frame,
$\dot{\vec{b}}^*$ - relative translational velocity of these two frames,
$2\boldsymbol{\Omega} \cdot (\vec{v}^* - \dot{\vec{b}}^*)$ - Coriolis acceleration,
$-\boldsymbol{\Omega} \cdot \boldsymbol{\Omega} \cdot (\vec{x}^* - \vec{b}^*)$ - centrifugal acceleration,
$\dot{\boldsymbol{\Omega}} \cdot (\vec{x}^* - \vec{b}^*)$ - Euler acceleration,
$\ddot{\vec{b}}^*$ - relative translational acceleration.

Among all Euclidean transformations, we can choose transformations that transform the acceleration in the objective way. In such a case, we have

$$\vec{a}^* = \boldsymbol{O} \cdot \vec{a} \quad \Leftrightarrow \quad \boldsymbol{\Omega} = \boldsymbol{0} , \ \ddot{\vec{b}}^* = \vec{0} \quad \Leftrightarrow \quad \vec{b}^*(t) = \vec{V} t + \vec{b}_0^* , \ \boldsymbol{O}(t) = \boldsymbol{O} , \tag{5.24}$$

where $\vec{V}$, $\vec{b}_0^*$ and $\boldsymbol{O}$ are time-independent. The change of frame defined by such constants,

$$\vec{x}^* = \boldsymbol{O} \cdot \vec{x} + \vec{V} t + \vec{b}_0^* , \tag{5.25}$$

is called the *Galilean transformation*. It means that the starred frame moved with a constant velocity with respect to the unstarred frame. Certainly, the acceleration is objective with respect to the Galilean transformation, whereas the velocity is not.

In contrast to the velocity field which is frame dependent (non-objective), the divergence of the velocity field is an objective scalar,

$$\operatorname{div}^* \vec{v}^* = \operatorname{div} \vec{v} . \tag{5.26}$$

To show it, we have

$$\operatorname{div}^* \vec{v}^* = \frac{\partial v^*_{k^*}}{\partial x^*_{k^*}} = \frac{\partial}{\partial x^*_{k^*}} \left[O_{k^*k} v_k + \Omega_{k^*k} (x^*_k - b^*_k) + \dot{b}^*_k \right] = O_{k^*k} \frac{\partial v_k}{\partial x^*_{k^*}} + \Omega_{k^*k} \frac{\partial x^*_k}{\partial x^*_{k^*}} = O_{k^*k} \frac{\partial v_k}{\partial x^*_{k^*}} + \Omega_{kk}$$

$$= O_{k^*k} \frac{\partial v_k}{\partial x^*_{k^*}} = O_{k^*k} \frac{\partial v_k}{\partial x_l} \frac{\partial x_l}{\partial x^*_{k^*}} = O_{k^*k} \frac{\partial v_k}{\partial x_l} O_{k^*l} = \delta_{kl} \frac{\partial v_k}{\partial x_l} = \frac{\partial v_k}{\partial x_k} = \operatorname{div} \vec{v} .$$

To study the effect of an observer transformation on the basic balance equations derived in Chapter 3, let us show that (i) the spatial gradient of an objective scalar is an objective vector, (ii) the spatial divergence of an objective vector is an objective scalar, and (iii) the spatial divergence of an objective tensor is an objective vector.

(i) Using (5.8) and $(5.16)_1$, we have

$$(\operatorname{grad}^* \lambda^*)_{k*} = \frac{\partial \lambda^*}{\partial x^*_{k^*}} = \frac{\partial \lambda}{\partial x_k} \frac{\partial x_k}{\partial x^*_{k^*}} = O_{k^*k} \frac{\partial \lambda}{\partial x_k} = O_{k^*k} \, (\operatorname{grad} \lambda)_k .$$

Multiplying by $\vec{i}_{k^*}$ and introducing a new vector grad $\lambda^* := \dfrac{\partial \lambda^*}{\partial x^*_{k^*}} \vec{i}_{k^*}$, we obtain

$$\text{grad}\,\lambda^* = \boldsymbol{O}(t) \cdot \text{grad}\,\lambda \; . \tag{5.27}$$

(ii) Next, from (5.8) and $(5.16)_2$, we have

$$\text{div}^*\, \vec{u}^* = \frac{\partial u^*_{k^*}}{\partial x^*_{k^*}} = \frac{\partial (O_{k^*l} u_l)}{\partial x_k} \frac{\partial x_k}{\partial x^*_{k^*}} = O_{k^*l} \frac{\partial u_l}{\partial x_k} O_{k^*k} = \delta_{kl} \frac{\partial u_l}{\partial x_k} = \frac{\partial u_k}{\partial x_k} = \text{div}\, \vec{u} \; . \tag{5.28}$$

(iii) And lastly, (5.8) and (5.12) give

$$(\text{div}^*\, \boldsymbol{a}^*)_{l^*} = \frac{\partial a^*_{k^*l^*}}{\partial x^*_{k^*}} = \frac{\partial (O_{k^*k} a_{kl} O_{l^*l})}{\partial x_m} \frac{\partial x_m}{\partial x^*_{k^*}} = O_{k^*k} O_{l^*l} \frac{\partial a_{kl}}{\partial x_m} O_{k^*m} = \delta_{km} O_{l^*l} \frac{\partial a_{kl}}{\partial x_m} = O_{l^*l} (\text{div}\, \boldsymbol{a})_l \; .$$

Multiplying by $\vec{i}_{l^*}$ and introducing a new vector div $\boldsymbol{a}^* := \dfrac{\partial a^*_{k^*l^*}}{\partial x^*_{k^*}} \vec{i}_{l^*}$, we obtain

$$\text{div}\, \boldsymbol{a}^* = \boldsymbol{O}(t) \cdot \text{div}\, \boldsymbol{a} \; . \tag{5.29}$$

The transformation rule for the deformation gradient is given by

$$\boldsymbol{F}^*(\vec{X}, t) = \boldsymbol{O}(t) \cdot \boldsymbol{F}(\vec{X}, t) \; , \tag{5.30}$$

where $\boldsymbol{F}^*(\vec{X}, t) := \chi^*_{k^*,K} (\vec{i}_{k^*} \otimes \vec{I}_K)$. To show it, we express the deformation gradient in the starred frame according to $(1.34)_1$ and substitute from (5.17):

$$F^*_{k^*K} = \frac{\partial \chi^*_{k^*}}{\partial X_K} = \frac{\partial}{\partial X_K} \Big(O_{k^*k}\, \chi_k + b^*_{k^*} \Big) = O_{k^*k} \frac{\partial \chi_k}{\partial X_K} = O_{k^*k} F_{kK} \; .$$

Multiplying by the tensor product $\vec{i}_{k^*} \otimes \vec{I}_K$, we obtain (5.30). Thus, the two-point deformation gradient tensor $\boldsymbol{F}$ is not an objective tensor. However, three columns of $\boldsymbol{F}$ (for $K = 1, 2, 3$) are objective vectors.

Let us verify that the jacobian J, the Green deformation tensor $\boldsymbol{C}$, the right stretch tensor $\boldsymbol{U}$ are all objective scalars, the rotation tensor $\boldsymbol{R}$ is an objective vector and the left stretch tensor $\boldsymbol{V}$, the Finger deformation tensor $\boldsymbol{b}$ and strain-rate tensor $\boldsymbol{d}$ are all objective tensors:

$$J^* = J, \qquad \boldsymbol{C}^* = \boldsymbol{C}, \qquad \boldsymbol{U}^* = \boldsymbol{U}, \tag{5.31}$$

$$\boldsymbol{R}^* = \boldsymbol{O}(t) \cdot \boldsymbol{R}, \tag{5.32}$$

$$\boldsymbol{V}^* = \boldsymbol{O}(t) \cdot \boldsymbol{V} \cdot \boldsymbol{O}^T(t), \qquad \boldsymbol{b}^* = \boldsymbol{O}(t) \cdot \boldsymbol{b} \cdot \boldsymbol{O}^T(t), \qquad \boldsymbol{d}^* = \boldsymbol{O}(t) \cdot \boldsymbol{d} \cdot \boldsymbol{O}^T(t) \; . \tag{5.33}$$

On contrary, the spatial velocity gradient $\boldsymbol{l}$ is not an objective tensor:

$$\boldsymbol{l}^* = \boldsymbol{O}(t) \cdot \boldsymbol{l} \cdot \boldsymbol{O}^T(t) + \boldsymbol{\Omega}(t) \; . \tag{5.34}$$

The proof is immediate by making use of (1.40), (1.48), (1.49), (1.53), (1.58), (2.12), (2.20), (5.11) and (5.30):

$$J^* = \det \boldsymbol{F}^* = \det(\boldsymbol{O} \cdot \boldsymbol{F}) = \det \boldsymbol{O} \det \boldsymbol{F} = \det \boldsymbol{F} = J,$$

$$
\begin{aligned}
&\boldsymbol{C}^* = (\boldsymbol{F}^*)^T \cdot \boldsymbol{F}^* = \boldsymbol{F}^T \cdot \boldsymbol{O}^T \cdot \boldsymbol{O} \cdot \boldsymbol{F} = \boldsymbol{F}^T \cdot \boldsymbol{F} = \boldsymbol{C},\\
&\boldsymbol{U}^* = \sqrt{\boldsymbol{C}^*} = \sqrt{\boldsymbol{C}} = \boldsymbol{U},\\
&\boldsymbol{R}^* = \boldsymbol{F}^* \cdot (\boldsymbol{U}^*)^{-1} = \boldsymbol{O} \cdot \boldsymbol{F} \cdot \boldsymbol{U}^{-1} = \boldsymbol{O} \cdot \boldsymbol{R},\\
&\boldsymbol{V}^* = \boldsymbol{R}^* \cdot \boldsymbol{U}^* \cdot (\boldsymbol{R}^*)^T = \boldsymbol{O} \cdot \boldsymbol{R} \cdot \boldsymbol{U} \cdot \boldsymbol{R}^T \cdot \boldsymbol{O}^T = \boldsymbol{O} \cdot \boldsymbol{V} \cdot \boldsymbol{O}^T,\\
&\boldsymbol{b}^* = \boldsymbol{F}^* \cdot (\boldsymbol{F}^*)^T = \boldsymbol{O} \cdot \boldsymbol{F} \cdot \boldsymbol{F}^T \cdot \boldsymbol{O}^T = \boldsymbol{O} \cdot \boldsymbol{b} \cdot \boldsymbol{O}^T,\\
&\boldsymbol{l}^* = (\boldsymbol{F}^*)^{\bullet} \cdot (\boldsymbol{F}^*)^{-1} = (\boldsymbol{O} \cdot \dot{\boldsymbol{F}} + \dot{\boldsymbol{O}} \cdot \boldsymbol{F})(\boldsymbol{O} \cdot \boldsymbol{F})^{-1} = \boldsymbol{O} \cdot \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{O}^T + \dot{\boldsymbol{O}} \cdot \boldsymbol{O}^T = \boldsymbol{O} \cdot \boldsymbol{l} \cdot \boldsymbol{O}^T + \boldsymbol{\Omega},\\
&\boldsymbol{d}^* = \frac{1}{2}(\boldsymbol{l}^* + \boldsymbol{l}^{*T}) = \frac{1}{2}(\boldsymbol{O} \cdot \boldsymbol{l} \cdot \boldsymbol{O}^T + \boldsymbol{O} \cdot \boldsymbol{l}^T \cdot \boldsymbol{O}^T + \boldsymbol{\Omega} + \boldsymbol{\Omega}^T) = \boldsymbol{O} \cdot \frac{1}{2}(\boldsymbol{l} + \boldsymbol{l}^T) \cdot \boldsymbol{O}^T = \boldsymbol{O} \cdot \boldsymbol{d} \cdot \boldsymbol{O}^T.
\end{aligned}
$$

5.3 Objective material time derivative

Let us now deal with the material time derivative of an objective scalar, an objective vector and an objective tensor. For an objective scalar λ, for which $\lambda^{**} = \lambda^* = \lambda$, it trivially holds that

$$\dot{\lambda}^* = \dot{\lambda} \ , \tag{5.35}$$

that is, the material time derivative of an objective scalar is again an objective scalar.

For an objective vector $\vec{u}$, for which $\vec{u}^* = \boldsymbol{O}(t) \cdot \vec{u}$, the material time derivative is

$$\dot{\vec{u}}^* = \boldsymbol{O} \cdot \dot{\vec{u}} + \dot{\boldsymbol{O}} \cdot \vec{u} = \boldsymbol{O} \cdot \dot{\vec{u}} + \dot{\boldsymbol{O}} \cdot \boldsymbol{O}^T \cdot \vec{u}^* \ ,$$

or, with the help of $\boldsymbol{\Omega} = \dot{\boldsymbol{O}} \cdot \boldsymbol{O}^T$, we have

$$\dot{\vec{u}}^* = \boldsymbol{O} \cdot \dot{\vec{u}} + \boldsymbol{\Omega} \cdot \vec{u}^* \ . \tag{5.36}$$

This means that the material time derivative of an objective vector is not an objective vector. There are a few possibilities where one can define time derivative of an objective vector to be again objective, and to obey a property of time derivative.

For example, the *Jaumann-Zaremba* or *corotational* time derivative of a vector $\vec{v}$ is defined as:

$$
\begin{aligned}
\frac{D_{\mathrm{Jau}}\vec{u}}{Dt} &:= \dot{\vec{u}} \ ,\\
\frac{D_{\mathrm{Jau}}\vec{u}^*}{Dt} &:= \dot{\vec{u}}^* - \boldsymbol{\Omega} \cdot \vec{u}^* \ .
\end{aligned} \tag{5.37}
$$

An immediate consequence of (5.36) is that

$$\frac{D_{\mathrm{Jau}}\vec{u}^*}{Dt} = \boldsymbol{O}(t) \cdot \frac{D_{\mathrm{Jau}}\vec{u}}{Dt} \ , \tag{5.38}$$

that is, the Jaumann time derivative of an objective vector is again objective vector.

Likewise, the material time derivative of an objective tensor $\boldsymbol{a}$, for which $\boldsymbol{a}^* = \boldsymbol{O}(t) \cdot \boldsymbol{a} \cdot \boldsymbol{O}^T(t)$, is

$$\dot{\boldsymbol{a}}^* = \boldsymbol{O} \cdot \dot{\boldsymbol{a}} \cdot \boldsymbol{O}^T + \dot{\boldsymbol{O}} \cdot \boldsymbol{a} \cdot \boldsymbol{O}^T + \boldsymbol{O} \cdot \boldsymbol{a} \cdot \dot{\boldsymbol{O}}^T = \boldsymbol{O} \cdot \dot{\boldsymbol{a}} \cdot \boldsymbol{O}^T + \dot{\boldsymbol{O}} \cdot \boldsymbol{O}^T \cdot \boldsymbol{a}^* \cdot \boldsymbol{O} \cdot \boldsymbol{O}^T + \boldsymbol{O} \cdot \boldsymbol{O}^T \cdot \boldsymbol{a}^* \cdot \boldsymbol{O} \cdot \dot{\boldsymbol{O}}^T \ ,$$

or, with the help of $\boldsymbol{\Omega} = \dot{\boldsymbol{O}} \cdot \boldsymbol{O}^T$, we have

$$\dot{\boldsymbol{a}}^* = \boldsymbol{O} \cdot \dot{\boldsymbol{a}} \cdot \boldsymbol{O}^T + \boldsymbol{\Omega} \cdot \boldsymbol{a}^* - \boldsymbol{a}^* \cdot \boldsymbol{\Omega} \ . \tag{5.39}$$

Hence, the material time derivative of an objective tensor is not objective.

The *Jaumann-Zaremba* or *corotational* time derivative of tensor $\boldsymbol{a}$ is defined as follows:

$$\begin{aligned} \frac{D_{\text{Jau}}\boldsymbol{a}}{Dt} &:= \dot{\boldsymbol{a}} \ , \\ \frac{D_{\text{Jau}}\boldsymbol{a}^*}{Dt} &:= \dot{\boldsymbol{a}}^* - \boldsymbol{\Omega} \cdot \boldsymbol{a}^* + \boldsymbol{a}^* \cdot \boldsymbol{\Omega} \ . \end{aligned} \tag{5.40}$$

An immediate consequence of (5.39) is

$$\frac{D_{\text{Jau}}\boldsymbol{a}^*}{Dt} = \boldsymbol{O}(t) \cdot \frac{D_{\text{Jau}}\boldsymbol{a}}{Dt} \cdot \boldsymbol{O}^T(t) \ , \tag{5.41}$$

that is, the Jaumann time derivative of an objective tensor is again an objective tensor.

The Oldroyd derivative is another possibility to introduce the objective time derivative of vectors and tensors. Let $\vec{u}$ and $\boldsymbol{a}$ be an objective vector and tensor, respectively. The *Oldroyd derivative* of $\vec{u}$ and $\boldsymbol{a}$ is defined by the respective formulae

$$\frac{D_{\text{Old}}\vec{u}}{Dt} := \dot{\vec{u}} - \boldsymbol{l} \cdot \vec{u} \ , \tag{5.42}$$

$$\frac{D_{\text{Old}}\boldsymbol{a}}{Dt} := \dot{\boldsymbol{a}} - \boldsymbol{l} \cdot \boldsymbol{a} - \boldsymbol{a} \cdot \boldsymbol{l}^T \ , \tag{5.43}$$

where $\boldsymbol{l}$ is the spatial velocity gradient defined by (2.13). The objectivity of the Oldroyd derivative of an objective vector follows from (5.20), (5.34), (5.36):

$$\begin{aligned} \frac{D_{\text{Old}}\vec{u}^*}{Dt} &= \dot{\vec{u}}^* - \boldsymbol{l}^* \cdot \vec{u}^* = \boldsymbol{O} \cdot \dot{\vec{u}} + \dot{\boldsymbol{O}} \cdot \vec{u} - (\boldsymbol{O} \cdot \boldsymbol{l} \cdot \boldsymbol{O}^T + \boldsymbol{\Omega}) \cdot \boldsymbol{O} \cdot \vec{u} \\ &= \boldsymbol{O} \cdot \dot{\vec{u}} + \dot{\boldsymbol{O}} \cdot \vec{u} - \boldsymbol{O} \cdot \boldsymbol{l} \cdot \boldsymbol{O}^T \cdot \boldsymbol{O} \cdot \vec{u} - \boldsymbol{\Omega} \cdot \boldsymbol{O} \cdot \vec{u} = \boldsymbol{O} \cdot \dot{\vec{u}} - \boldsymbol{O} \cdot \boldsymbol{l} \cdot \vec{u} \ . \end{aligned}$$

Hence,

$$\frac{D_{\text{Old}}\vec{u}^*}{Dt} = \boldsymbol{O}(t) \cdot \frac{D_{\text{Old}}\vec{u}}{Dt} \ . \tag{5.44}$$

Likewise, making use of (5.20), (5.34), (5.39), we have

$$\begin{aligned} \frac{D_{\text{Old}}\boldsymbol{a}^*}{Dt} &= \dot{\boldsymbol{a}}^* - \boldsymbol{l}^* \cdot \boldsymbol{a}^* - \boldsymbol{a}^* \cdot \boldsymbol{l}^{*T} \\ &= \boldsymbol{O} \cdot \dot{\boldsymbol{a}} \cdot \boldsymbol{O}^T + \dot{\boldsymbol{O}} \cdot \boldsymbol{a} \cdot \boldsymbol{O}^T + \boldsymbol{O} \cdot \boldsymbol{a} \cdot \dot{\boldsymbol{O}}^T \\ &- (\boldsymbol{O} \cdot \boldsymbol{l} \cdot \boldsymbol{O}^T + \boldsymbol{\Omega}) \cdot \boldsymbol{O} \cdot \vec{a} \cdot \boldsymbol{O}^T - \boldsymbol{O} \cdot \vec{a} \cdot \boldsymbol{O}^T \cdot (\boldsymbol{O} \cdot \boldsymbol{l}^T \cdot \boldsymbol{O}^T + \boldsymbol{\Omega}^T) \\ &= \boldsymbol{O} \cdot \dot{\boldsymbol{a}} \cdot \boldsymbol{O}^T - \boldsymbol{O} \cdot \boldsymbol{l} \cdot \boldsymbol{a} \cdot \boldsymbol{O}^T - \boldsymbol{O} \cdot \boldsymbol{a} \cdot \boldsymbol{l}^T \cdot \boldsymbol{O}^T \ . \end{aligned}$$

Hence,

$$\frac{D_{\text{Old}}\boldsymbol{a}^*}{Dt} = \boldsymbol{O}(t) \cdot \frac{D_{\text{Old}}\boldsymbol{a}}{Dt} \cdot \boldsymbol{O}^T(t) \ . \tag{5.45}$$