6. CONSTITUTIVE EQUATIONS

6.1 The need for constitutive equations

Basic principles of continuum mechanics, namely, conservation of mass, balance of momenta, and conservation of energy, discussed in Chapter 4, lead to the fundamental equations:

\[
\frac{\partial \rho}{\partial t} + \text{div} (\rho \vec{v}) = 0 ,
\]

\[
\text{div} \vec{t} + \rho \vec{f} = \rho \frac{D\vec{v}}{Dt}, \quad \vec{t} = \vec{t}^T ,
\]

\[
\rho \frac{D\varepsilon}{Dt} = \vec{t} : \vec{d} - \text{div} \vec{q} + \rho h .
\]

In total, they constitute 5 independent equations (one for mass, three for linear momentum and one for energy) for 15 unknown field variables, namely,

- mass density \( \rho \),
- velocity \( \vec{v} \),
- Cauchy’s stress tensor \( \vec{t} \),
- internal energy \( \varepsilon \),
- heat flux \( \vec{q} \),
- temperature \( \theta \)

provided that body forces \( \vec{f} \) and distribution of heat sources \( h \) are given. Clearly, the foregoing basic equations are not adequate for the determination of these unknowns except for some trivial situations, for example, rigid body motions in the absence of heat conduction. Hence, 10 additional equations must be supplied to make the problem well-posed.

In the derivation of the equations (6.1) to (6.3) no differentiation has been made between various types materials. It is therefore not surprising that the foregoing equations are not sufficient to explain fully the motions of materials having various type of physical properties. The character of the material is brought into the formulation through the so-called constitutive equations, which specify the mechanical and thermal properties of particular materials based upon their internal constitution. Mathematically, the usefulness of these constitutive equations is to describe the relationships among the kinematic, mechanical, and thermal field variables and to permit the formulations of well-posed problems of continuum mechanics. Physically, the constitutive equations define various idealized materials which serve as models for the behavior of real materials. However, it is not possible to write one equation capable of representing a given material over its entire range of application, since many materials behave quite differently under changing levels of loading, such as elastic-plastic response due to increasing stress. Thus, in this sense it is perhaps better to think of constitutive equations as representative of a particular behavior rather than of a particular material.

6.2 Formulation of thermomechanical constitutive equations

In this text we deal with the constitutive equations of thermomechanical materials. The study of the chemical changes and electromagnetic effects are excluded. A large class of materials does
not undergo chemical transition or produce appreciable electromagnetic effects when deformed. However, the deformation and motion generally produce heat. Conversely, materials subjected to thermal changes deform and flow. The effect of thermal changes on the material behavior depends on the range and severity of such changes.

The thermomechanical constitutive equations are relations between a set of thermomechanical variables. They may be expressed as an implicit tensor-valued functional $\mathcal{R}$ of 15 unknown field variables:

$$
\mathcal{R} \left[ \varrho(\vec{X}', \tau), \vec{\chi}(\vec{X}', \tau), \theta(\vec{X}', \tau), \vec{t}(\vec{X}', \tau), \vec{q}(\vec{X}', \tau), \varepsilon(\vec{X}', \tau), \vec{X} \right] = 0,
$$

(6.4)

where $\tau$ are all past times and $t$ is the present time. The constraints $\vec{X}' \in \mathcal{B}$ and $\tau \leq t$ express the principle of determinism postulating that the present state of the thermomechanical variables at a material point $\vec{X}$ of the body $\mathcal{B}$ at time $t$ is uniquely determined by the past history of the motion and the temperature of all material points of the body $\mathcal{B}$. The principle of determinism is a principle of exclusions. It excludes the dependence of the material behavior on any point outside the body and any future events.

We shall restrict the functional in (6.4) to be of a type that does not change with time, that is, that does not depend on the present time $t$ explicitly but only implicitly via thermomechanical variables. Such a functional is invariant with respect to translation in time. The materials described by (6.4) possess time-independent thermomechanical property.

The constitutive functional $\mathcal{R}$ describes the material property of a given material particle $\mathcal{X}$ with the position $\vec{X}$. The functional form may, in general, be different for different particles and $\mathcal{R}$ may thus change with the change of position within the body $\mathcal{B}$; such a material is called heterogeneous. If functional $\mathcal{R}$ is independent of $\vec{X}$, material is homogeneous.

For a simple material (see the next section), the implicit functional equation (6.4) is supposed to be solved uniquely for the present values of thermomechanical variables. In this case, the implicit functional equation (6.4) is replaced by a set of explicit functional equations:

$$
t(\vec{X}, t) = \mathcal{F} \left[ \varrho(\vec{X}', \tau), \vec{\chi}(\vec{X}', \tau), \theta(\vec{X}', \tau), \vec{X} \right],
$$

$$
\vec{q}(\vec{X}, t) = \mathcal{Q} \left[ \varrho(\vec{X}', \tau), \vec{\chi}(\vec{X}', \tau), \theta(\vec{X}', \tau), \vec{X} \right],
$$

$$
\varepsilon(\vec{X}, t) = \mathcal{E} \left[ \varrho(\vec{X}', \tau), \vec{\chi}(\vec{X}', \tau), \theta(\vec{X}', \tau), \vec{X} \right],
$$

(6.5)

where $\mathcal{F}$, $\mathcal{Q}$ and $\mathcal{E}$ are respectively tensor-valued, vector-valued and scalar-valued functionals. Note that all constitutive functionals $\mathcal{F}$, $\mathcal{Q}$ and $\mathcal{E}$ are assumed to depend on the same set of variables $\varrho(\vec{X}', \tau)$, $\vec{\chi}(\vec{X}', \tau)$, $\theta(\vec{X}', \tau)$ and $\vec{X}$. This is known as the principle of equipresence.

However, the implicit functional equation (6.4) need not be of such a nature as to determine $t$, $\vec{q}$ and $\varepsilon$ at $(\vec{X}, t)$ explicitly. For instance, the stress at $(\vec{X}, t)$ may depend not only on the motion and temperature at all other points of the body but also on the histories of the stress, the heat flux and the internal energy. Various types of approximations of (6.4) exist in which the dependence on $t(\vec{X}', \tau)$ is replaced by the history of various order of stress rates, heat rates, etc.
For example, in a special case the constitutive equation (6.4) may be written explicitly for the stress rates $\dot{t}$ at $(\vec{X}, t)$:

$$\dot{t}(\vec{X}, t) = \mathcal{F}_{\vec{X}^\prime \in \mathcal{B}} \left[ t(\vec{X}, t), q(\vec{X}, \tau), \vec{X}^\prime(\vec{X}, t), \theta(\vec{X}^\prime, \tau) \right]. \quad (6.6)$$

More generally, we may have

$$t^{(p)}(\vec{X}, t) = \mathcal{F}_{\vec{X}^\prime \in \mathcal{B}} \left[ t^{(p-1)}(\vec{X}, t), t^{(p-2)}(\vec{X}, t), \ldots, t(\vec{X}, t), q(\vec{X}), \vec{X}^\prime(\vec{X}, \tau), \theta(\vec{X}^\prime, \tau) \right], \quad (6.7)$$

which involves the stress rates up to the $p$th order at $(\vec{X}, t)$. This type of generalization is, for instance, needed to interpret creep data.

Except the Maxwell viscoelastic solid, we shall consider the explicit constitutive equations (6.5). Together with 5 basic balance equations (6.1)–(6.3) they form 15 equations for 15 unknowns. Since $t, \vec{q},$ and $\varepsilon$ are expressed explicitly in (6.5), it is, in principle, possible to eliminate these variables in (6.1)–(6.3). Then we obtain 5 equations (the so-called field equations of thermomechanics) for 5 unknown field variables: mass density $\rho$, velocity $\vec{v}$, and temperature $\theta$.

We now proceed to deduce the consequence of additional restrictions on the functionals $\mathcal{F}, \mathcal{Q}$ and $\mathcal{E}$. Since the procedure is similar for all functionals, for the sake of brevity, we carry out the analysis only for stress functional $\mathcal{F}$. The results for $\mathcal{Q}$ and $\mathcal{E}$ are then written down immediately by analogy.

### 6.3 Simple materials

The constitutive functionals $\mathcal{F}, \mathcal{Q}$ and $\mathcal{E}$ are subject to another fundamental principle, the **principle of local action** postulating that the motion and the temperature at distant material points from $\vec{X}$ does not affect appreciably the stress, the heat flux and the internal energy at $\vec{X}$. Suppose that the functions $q(\vec{X}^\prime, \tau), \vec{x}(\vec{X}^\prime, \tau)$ and $\theta(\vec{X}^\prime, \tau)$ admit Taylor series expansion about $\vec{X}$ for all $\tau < t$. According to the principle of local action, a relative deformation of the neighborhood of material point $\vec{X}$ is permissible to approximate only by the first-order gradient

$$
\begin{align*}
q(\vec{X}^\prime, \tau) &\approx q(\vec{X}, \tau) + \text{Grad} q(\vec{X}, \tau) \cdot d\vec{X}, \\
\vec{x}(\vec{X}^\prime, \tau) &\approx \vec{x}(\vec{X}, \tau) + \mathcal{F}(\vec{X}, \tau) \cdot d\vec{X}, \\
\theta(\vec{X}^\prime, \tau) &\approx \theta(\vec{X}, \tau) + \text{Grad} \theta(\vec{X}, \tau) \cdot d\vec{X},
\end{align*}
\quad (6.8)
$$

where $\mathcal{F}(\vec{X}, \tau)$ is the deformation gradient tensor at $\vec{X}$ and time $\tau$, and $d\vec{X} = \vec{X}^\prime - \vec{X}$. Since the relative motion and temperature history of an infinitesimal neighborhood of $\vec{X}$ is completely determined by the history of density, deformation and temperature gradients at $\vec{X}$, the stress $t(\vec{X}, t)$ must be determined by the history of $\text{Grad} q(\vec{X}, \tau), \mathcal{F}(\vec{X}, \tau)$ and $\text{Grad} \theta(\vec{X}, \tau)$ for $\tau \leq t$. Such materials are called **simple materials**. We also note that if we retain higher-order gradients in (6.8), then we obtain nonsimple materials of various classes. For example, by including the second-order gradients into argument of $\mathcal{F}$ we get the theory of **couple stress**. In other words, the behavior of the material point $\vec{X}$ within a simple material is not affected by the histories of the distant points from $\vec{X}$. To any desired degree of accuracy, the whole configuration of a sufficiently small neighborhood of the material point $\vec{X}$ is determined by the history of $\text{Grad} q(\vec{X}, \tau), \mathcal{F}(\vec{X}, \tau)$
and \( \text{Grad} \theta(\vec{X}, \tau) \), and we may say that the stress \( t(\vec{X}, t) \), which was assumed to be determined by the local configuration, is completely determined by \( \text{Grad} \rho(\vec{X}, \tau), F(\vec{X}, \tau) \) and \( \text{Grad} \theta(\vec{X}, \tau) \). That is, the general constitutive equation (6.5) reduces to the form

\[
t(\vec{X}, t) = \mathcal{F} \left[ \rho(\vec{X}, \tau), \text{Grad} \rho(\vec{X}, \tau), \chi(\vec{X}, \tau), F(\vec{X}, \tau), \theta(\vec{X}, \tau), \text{Grad} \theta(\vec{X}, \tau), \vec{X} \right]. \tag{6.9}
\]

Given the deformation gradient \( F \), the density in the reference configuration is expressed through the continuity equation (4.60) as

\[
\rho(\vec{X}, \tau) = \frac{\rho_0(\vec{X})}{\det F(\vec{X}, \tau)}. \tag{6.10}
\]

We can thus drop the argument \( \rho(\vec{X}, \tau) \) in the functional \( \mathcal{F} \) since \( \rho(\vec{X}, \tau) \) is expressible in terms of \( F(\vec{X}, \tau) \) (the factor \( \rho_0(\vec{X}) \) is a fixed expression – not dependent on time – for a given reference configuration). Moreover, by applying the gradient operator on (6.10), the term \( \text{Grad} \rho(\vec{X}, \tau) \) can be expressed in terms of \( \text{Grad} F(\vec{X}, \tau) \). For a simple material, however, \( \text{Grad} F \) can be neglected with respect to \( F \). In summary, the principle of local action applied for a simple material leads to the following constitutive equation:

\[
t(\vec{X}, t) = \mathcal{F} \left[ \chi(\vec{X}, \tau), F(\vec{X}, \tau), \theta(\vec{X}, \tau), \vec{G}_\theta(\vec{X}, \tau), \vec{X} \right], \tag{6.11}
\]

where \( \vec{G}_\theta \) stands for the temperature gradient, \( \vec{G}_\theta := \text{Grad} \theta \).

### 6.4 Material objectivity

The change of a reference frame to another reference frame was studied in Chapter 5, where the quantities such as velocity, acceleration and other kinematic quantities were transformed from a fixed reference frame to a moving reference frame. The most general form of a transformation between two reference frames moving against each other is the observer transformation discussed in Section 5.1.

We now will specify the behavior under the observer transformation of the fields which represent the primitive concepts of mass, force, internal energy and heating. Of these the density \( \rho \), the stress vector \( \vec{t}(n) \), the internal energy \( \varepsilon \) and the heat flux \( \vec{h}(\vec{n}) := \vec{q} \cdot \vec{n} \), being associated with the internal circumstances of a material body, are expected to appear the same to equivalent observers. They are accordingly taken to be objective, an assumption which may properly be viewed as a part of the principle of objectivity. From this assumption, we will show that the Cauchy stress tensor \( \vec{t} \) and the heat flux vector \( \vec{q} \) are objective. The objectivity of \( \vec{t}(n) \) means that

\[
[t(n^*)]_{k^*} = O_{k^*k}[t(n)]_k, \tag{6.12}
\]

or, by the Cauchy stress formula (3.15), \( [t(n^*)]_k = n_l t_{lk} \), it holds

\[
n^*_l t^*_{lk} = O_{k^*k} n_l t_{lk}. \tag{6.13}
\]

Making use of the objectivity of the normal vector \( \vec{n} \), \(^1\) that is \( n^*_l = O_{l^*l} n_l \) leads to

\[
O_{l^*l} n_l t^*_{lk} = O_{k^*k} n_l t_{lk}. \tag{6.14}
\]

\(^1\)The normal vector to a surface may be defined as the gradient of a scalar function. From (5.27) it then follows that \( \vec{n} \) is an objective vector.
Introducing a new tensor $t^* := t^*_{k^*}((\tilde{k^*} \otimes \tilde{n^*})$, we have
\[ \tilde{n} \cdot (\bar{O}^T t^* - t \cdot \bar{O}^T) = \bar{0}, \] (6.15)
which must hold for all surface passing through a material point. Hence $\bar{O}^T t^* - t \cdot \bar{O}^T = 0$. With
the orthogonality property of $\bar{O}$, we finally obtain
\[ t^* = \bar{O}(t) \cdot t \cdot \bar{O}^T(t), \] (6.16)
which shows that the Cauchy stress tensor is an objective tensor. Likewise, the objectivity of $\bar{q} \cdot \tilde{n}$ implies that the heat flux vector $\bar{q}$ is an objective vector,
\[ \bar{q}^* = \bar{O}(t) \cdot \bar{q}. \] (6.17)

Let us verify that the referential heat flux vector $\bar{G}_\theta$ and the second Piola-Kirchhoff stress tensor $T^{(2)}$ are both objective scalars:
\[ \bar{G}^*_{\theta*} = \bar{G}_\theta, \quad T^{(2)*} = T^{(2)}. \] (6.18)

The proof is immediate by making use of (3.26), (5.30), (5.31) and (6.16):
\[ \bar{G}^*_{\theta*} = \text{Grad} \theta^* = \text{Grad} \theta = \bar{G}_\theta, \]
\[ T^{(2)*} = J^*(F^*)^{-1} \cdot t^* \cdot (F^*)^{-T} = JF^{-1} \cdot Q^T \cdot Q \cdot t \cdot Q^T \cdot Q \cdot F^{-T} = JF^{-1} \cdot t \cdot F^{-T} = T^{(2)}. \]

The constitutive functionals are subject to yet another fundamental principle. It has become practical evidence that there are no known cases in which constitutive equations are frame dependent. The postulate of the indifference of the constitutive equations against observer transformations is called the \textbf{principle of material objectivity}. This principle states that a constitutive equation must be form-invariant under rigid motions of the spatial frame or, in other words, the material properties cannot depend on the motion of an observer. To express this principle mathematically, let us write the constitutive equation (6.11) in the unstarred and starred frames:
\[ \bar{t}(\bar{X}, t) = \mathcal{F}_{\tau \leq t} \left[ \chi(\bar{X}, \tau), \bar{F}(\bar{X}, \tau), \theta(\bar{X}, \tau), \bar{G}_\theta(\bar{X}, \tau), \bar{X} \right], \]
\[ \bar{t}^*(\bar{X}, t) = \mathcal{F}^*_{\tau \leq t} \left[ \chi^*(\bar{X}, \tau), \bar{F}^*(\bar{X}, \tau), \theta^*(\bar{X}, \tau), \bar{G}^*_\theta(\bar{X}, \tau), \bar{X} \right], \] (6.19)
where $\chi(\bar{X}, t)$ and $\chi^*(\bar{X}, t)$ are related by (5.17). In general, the starred and unstarred functionals $\mathcal{F}$ and $\mathcal{F}^*$ may differ, but the principle of material objectivity requires that the form of the constitutive functional $\mathcal{F}$ must be the same under any two rigid motions of spatial frame. Mathematically, no star is attached to the functional $\mathcal{F}$:
\[ \mathcal{F}[ \cdot ] = \mathcal{F}^* [ \cdot ], \] (6.20)
where the arguments are those of (6.19). \footnote{This is a significant advantage of the Lagrangian description, since material properties are always associated with a given material particle $x$ with the position vector $\bar{X}$, while, in the Eulerian description, various material particles may pass through a given spatial position $\bar{x}$.} Written it in the form (6.16), we have
\[ \bar{O}(t) \cdot \mathcal{F}_{\tau \leq t} \left[ \chi(\bar{X}, \tau), \bar{F}(\bar{X}, \tau), \theta(\bar{X}, \tau), \bar{G}_\theta(\bar{X}, \tau), \bar{X} \right] \cdot \bar{O}^T(t) \]
\[ = \mathcal{F}_{\tau \leq t} \left[ \chi^*(\bar{X}, \tau), \bar{F}^*(\bar{X}, \tau), \theta^*(\bar{X}, \tau), \bar{G}^*_\theta(\bar{X}, \tau), \bar{X} \right]. \]
Taking into account the transformation relations (5.17), (5.30) and (6.18), the restriction placed on constitutive functional $\mathcal{F}$ is

$$O(t) \cdot \mathcal{F}_{\tau \leq t} \left[ \chi(\bar{X}, \tau), F(\bar{X}, \tau), \theta(\bar{X}, \tau), G_\theta(\bar{X}, \tau), \bar{X} \right] \cdot O^T(t)$$

$$= \mathcal{F}_{\tau \leq t} \left[ O(\tau) \cdot \chi(\bar{X}, \tau) + \bar{b}^*(\tau), O(\tau) \cdot F(\bar{X}, \tau), \theta(\bar{X}, \tau), G_\theta(\bar{X}, \tau), \bar{X} \right] .$$

This relation must hold for any arbitrary orthogonal tensor-valued functions $O(t)$ and any arbitrary vector-valued function $\bar{b}^*(t)$.

In particular, let us consider a rigid translation of the spatial frame such that the origin of the spatial frame moves with the material point $\bar{X}$:

$$O(\tau) = I , \quad \bar{b}^*(\tau) = -\chi(\bar{X}, \tau) .$$

This means that the spatial frame of reference is translated so that the material point $\bar{X}$ at any time $\tau$ remains at the origin. From (5.17) it follows that $\chi^0(\bar{X}, \tau) = 0$ and (6.21) becomes

$$\mathcal{F}_{\tau \leq t} \left[ \chi(\bar{X}, \tau), F(\bar{X}, \tau), \theta(\bar{X}, \tau), G_\theta(\bar{X}, \tau), \bar{X} \right] = \mathcal{F}_{\tau \leq t} \left[ 0, F(\bar{X}, \tau), \theta(\bar{X}, \tau), G_\theta(\bar{X}, \tau), \bar{X} \right] .$$

Thus the stress at the material point $\bar{X}$ and time $t$ cannot depend explicitly on the history of motion of this point. It also implies that velocity and acceleration and all other higher time derivatives of the motion have no influence on the material laws. The translation of the frame, $\bar{b}^*$, is thus irrelevant. Consequently, the general constitutive equation (6.11) reduces to the form

$$t(\bar{X}, t) = \mathcal{F}_{\tau \leq t} \left[ F(\bar{X}, \tau), \theta(\bar{X}, \tau), G_\theta(\bar{X}, \tau), \bar{X} \right] .$$

The restriction (6.21) is now reduced to the form

$$O(t) \cdot \mathcal{F}_{\tau \leq t} \left[ F(\bar{X}, \tau), \theta(\bar{X}, \tau), G_\theta(\bar{X}, \tau), \bar{X} \right] \cdot O^T(t) = \mathcal{F}_{\tau \leq t} \left[ O(\tau) \cdot F(\bar{X}, \tau), \theta(\bar{X}, \tau), G_\theta(\bar{X}, \tau), \bar{X} \right] ,$$

which must hold for all orthogonal tensor-valued functions $O(t)$ and all invertible deformation processes $F(\bar{X}, t)$.

### 6.5 Reduction by polar decomposition

The condition (6.25) will now be used to reduce the constitutive equation (6.24). Let us recall the polar decomposition (1.47) of the deformation gradient $F(\bar{X}, t) = R(\bar{X}, t) \cdot U(\bar{X}, t)$ into a rotation tensor $R$ and the right stretch tensor $U = \sqrt{\mathcal{C}} = \sqrt{F^T \cdot F}$. We now make a special choice for the orthogonal tensor $O(t)$ in (6.25). For any fixed reference place $\bar{X}$, we put $O(t) = R^T(\bar{X}, t)$ for all $t$. This special choice of $O(t)$ in (6.25) yields:

$$R^T(\bar{X}, t) \cdot \mathcal{F}_{\tau \leq t} \left[ F(\bar{X}, \tau), \theta(\bar{X}, \tau), G_\theta(\bar{X}, \tau), \bar{X} \right] \cdot R(\bar{X}, t)$$

$$= \mathcal{F}_{\tau \leq t} \left[ R^T(\bar{X}, \tau) \cdot F(\bar{X}, \tau), \theta(\bar{X}, \tau), G_\theta(\bar{X}, \tau), \bar{X} \right] ,$$

6
which with $U = R^T \cdot F$ reduces to

$$\mathcal{F} \left[ \mathcal{F}(\tilde{X}, \tau), \theta(\tilde{X}, \tau), \tilde{G}_\theta(\tilde{X}, \tau), \tilde{X} \right] = R(\tilde{X}, t) \cdot \mathcal{F} \left[ U(\tilde{X}, \tau), \theta(\tilde{X}, \tau), \tilde{G}_\theta(\tilde{X}, \tau), \tilde{X} \right] \cdot R^T(\tilde{X}, t) .$$ \hspace{1cm} (6.27)

This reduced form has been obtained for a special choice of $O(t)$ in the principle of material objectivity (6.25). This means that (6.27) is a necessary relation for satisfying the principle of material objectivity. Now, we shall prove that (6.27) is also a sufficient relation for satisfying this principle. Suppose, that $\mathcal{F}$ is of the form (6.27) and consider an arbitrary orthogonal tensor history $O(t)$. Since the polar decomposition of $O \cdot F$ is $(O \cdot R) \cdot U$, (6.27) for $O \cdot F$ reads

$$\mathcal{F} \left[ O(\tau) \cdot F(\tilde{X}, \tau), \theta(\tilde{X}, \tau), \tilde{G}_\theta(\tilde{X}, \tau), \tilde{X} \right] = O(t) \cdot R(\tilde{X}, t) \cdot \mathcal{F} \left[ U(\tilde{X}, \tau), \theta(\tilde{X}, \tau), \tilde{G}_\theta(\tilde{X}, \tau), \tilde{X} \right] \cdot [O(t) \cdot R(\tilde{X}, t)]^T ,$$

which in view of (6.27) reduces to (6.25), so that (6.25) is satisfied. Therefore the reduced form (6.27) is necessary and sufficient to satisfy the principle of material objectivity.

We have proved that the constitutive equation of a simple material may be put into the form

$$t(\tilde{X}, t) = R(\tilde{X}, t) \cdot \mathcal{F} \left[ U(\tilde{X}, \tau), \theta(\tilde{X}, \tau), \tilde{G}_\theta(\tilde{X}, \tau), \tilde{X} \right] \cdot R^T(\tilde{X}, t) .$$ \hspace{1cm} (6.28)

A constitutive equation of this kind, in which the functionals are not subject to any further restriction, is called a reduced form. The result (6.28) shows that while the stretch history of a simple material may affect its present stress, past rotations have no effect at all. The present rotation enters (6.28) explicitly.

There are many other reduced forms for the constitutive equation of a simple material. Replacing the stretch $U$ by the Green deformation tensor $C$, $U = \sqrt{C}$, and denoting the functional $\mathcal{F}(\sqrt{C}, \cdots)$ again as $\mathcal{F}(C, \cdots)$, we obtain

$$t(\tilde{X}, t) = R(\tilde{X}, t) \cdot \mathcal{F} \left[ C(\tilde{X}, \tau), \theta(\tilde{X}, \tau), \tilde{G}_\theta(\tilde{X}, \tau), \tilde{X} \right] \cdot R^T(\tilde{X}, t) .$$ \hspace{1cm} (6.29)

Likewise, expressing rotation $R$ through the deformation gradient, $R = F \cdot U^{-1}$, and replacing the stretch $U$ by Green’s deformation tensor $C$, equation (6.29), after introducing a new functional $\tilde{\mathcal{F}}$ of the deformation history $C$, can be put into another reduced form

$$t(\tilde{X}, t) = F(\tilde{X}, t) \cdot \tilde{\mathcal{F}} \left[ C(\tilde{X}, \tau), \theta(\tilde{X}, \tau), \tilde{G}_\theta(\tilde{X}, \tau), \tilde{X} \right] \cdot F^T(\tilde{X}, t) .$$ \hspace{1cm} (6.30)

We should emphasize that $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are materially objective functionals.

The principle of material objectivity applied to the constitutive equations (6.5) for the heat flux and the internal energy in analogous way as for the stress results in the following reduced forms:

$$\tilde{q}(\tilde{X}, t) = R(\tilde{X}, t) \cdot \tilde{Q} \left[ C(\tilde{X}, \tau), \theta(\tilde{X}, \tau), \tilde{G}_\theta(\tilde{X}, \tau), \tilde{X} \right] , \hspace{1cm} (6.31)$$

$$\tilde{\varepsilon}(\tilde{X}, t) = \tilde{\mathcal{E}} \left[ C(\tilde{X}, \tau), \theta(\tilde{X}, \tau), \tilde{G}_\theta(\tilde{X}, \tau), \tilde{X} \right] .$$
Another useful reduced form may be obtained if the second Piola-Kirchhoff stress tensor $T^{(2)}$ defined by (3.26)_,

$$T^{(2)} = (\det F) \cdot F^{-1} \cdot t \cdot F^{-T}, \quad (6.32)$$

is used in the constitutive equation instead of the Cauchy stress tensor $t$. With

$$J = \det F = \det (R \cdot U) = \det U = \sqrt{\det C}, \quad (6.33)$$

and defining a new functional $G$ of the deformation history $C$, the constitutive equation (6.30) can be rewritten for the second Piola-Kirchhoff stress tensor as

$$T^{(2)}(\vec{X}, t) = G_{\tau \leq t} [C(\vec{X}, \tau), \theta(\vec{X}, \tau), \vec{G}_\theta(\vec{X}, \tau), \vec{X}] \quad (6.34)$$

According to this result, the second Piola-Kirchhoff stress tensor depends only on the Green deformation tensor and not on the rotation. Likewise, the constitutive functional for the heat flux vector in the reference configuration,

$$\vec{Q} = JF^{-1} \cdot \vec{q}, \quad (6.35)$$

is of the form

$$\vec{Q}(\vec{X}, t) = \vec{Q}_{\tau \leq t} [C(\vec{X}, \tau), \theta(\vec{X}, \tau), \vec{G}_\theta(\vec{X}, \tau), \vec{X}] \quad (6.36)$$
6.6 Kinematic constraints

A condition of kinematic constraint is a geometric restriction on the set of all motions which are possible for a material body. A condition of constraint can be defined as a restriction on the deformation gradients:

$$\lambda(\mathbf{F}(t)) = 0 , \quad (6.37)$$

where to lighten notations in this section we drop the place $X$ from the notation as in $\mathbf{F}(t) \equiv \mathbf{F}(X,t)$. The requirement that kinematic constraints shall be materially objective implies that equation (6.57) has to be replaced by

$$\lambda(\mathbf{C}(t)) = 0 . \quad (6.38)$$

In the context of kinematic constraints, the principle of determinism must be modified; it postulates that only a part of the stress tensor is related to the history of the deformation. Thus the stress tensor (6.24) obtains an additional term

$$\mathbf{t}(t) = \mathbf{\pi}(t) + \mathbf{F} \left[ \mathbf{F}(\theta), \theta(\tau), \mathbf{G}_\theta(\tau) \right]. \quad (6.39)$$

The tensor $\mathbf{t} - \mathbf{\pi}$ is called the *determinate stress* because it is uniquely determined by the motion. The tensor $\mathbf{\pi}$ is called *indeterminate stress* and represents the *reaction stress* produced by the kinematic constraint (6.38). It is not determined by the motion. In analogy with analytical mechanics, it is assumed that the reaction stress does not perform any work, that is the stress power vanishes for all motions compatible with the constraint condition (6.38):

$$\mathbf{\pi}(t) : \mathbf{d}(t) = 0 , \quad (6.40)$$

where $\mathbf{d}$ is the strain-rate tensor defined by (2.20)$_1$. 

To derive an alternative form, we modify the constitutive equation (6.34) for the second Piola-Kirchhoff stress tensor as

$$\mathbf{T}^{(2)}(t) = \mathbf{\Pi}(t) + \mathbf{G} \left[ \mathbf{C}(\tau), \theta(\tau), \mathbf{G}_\theta(\tau) \right]. \quad (6.41)$$

If we take into account (2.23) and (3.26)$_2$, the double-dot product $\mathbf{\pi} : \mathbf{d}$ can be expressed in terms of the reaction stress $\mathbf{\Pi}$ as

$$\mathbf{\pi} : \mathbf{d} = \frac{1}{J} (\mathbf{F} \cdot \mathbf{\Pi} \cdot \mathbf{F}^T) : \frac{1}{2} (\mathbf{F}^{-T} \cdot \dot{\mathbf{C}} \cdot \mathbf{F}^{-1}) = \frac{1}{2J} (\mathbf{\Pi} : \dot{\mathbf{C}}).$$

Hence, the constraint (6.40) takes the referential form

$$\mathbf{\Pi}(t) : \dot{\mathbf{C}}(t) = 0 . \quad (6.42)$$

This says that the reaction stress $\mathbf{\Pi}$ has no power to work for all motions compatible with the constraints (6.38). This compatibility can be evaluated in a more specific form. Differentiating (6.38) with respect to time, we obtain

$$\frac{d\lambda(\mathbf{C}(t))}{d\mathbf{C}_{KL}} \mathbf{C}_{KL} = 0 \quad \text{or, symbolically,} \quad \frac{d\lambda(\mathbf{C}(t))}{d\mathbf{C}} : \dot{\mathbf{C}} = 0 . \quad (6.43)$$
This shows that the normal \( d\lambda/dC \) to the surface \( \lambda(C) = \text{const.} \) is orthogonal to all strain rates \( \dot{C} \) which are allowed by the constraint condition (6.38). Equation (6.43) suggests that just the same orthogonality holds for the reaction stress \( \Pi \), whence follows that \( \Pi \) must be parallel to \( d\lambda/dC \):

\[
\Pi(t) = \alpha(t) \frac{d\lambda(C(t))}{dC}. \tag{6.44}
\]

Here, the factor \( \alpha \) is left undetermined and must be regarded as an independent field variable in the balance law of linear momentum.

The same arguments apply if simultaneously more constraints are specified. For \( \lambda_i(C(t)) = 0 \), \( i = 1, 2, \ldots \), we have

\[
T^{(2)}(t) = \sum_i \Pi_i(t) + G_{\tau \leq t} [C(\tau), \theta(\tau), \text{Grad} \theta(\tau)] \tag{6.46}
\]

with

\[
\Pi_i(t) = \alpha_i(t) \frac{d\lambda_i(C(t))}{dC}. \tag{6.47}
\]

As an example we investigate the constraint of incompressibility. If a body is assumed to be incompressible, only volume-preserving motions are possible. It also means that the mass density remains unchanged, that is,

\[
J = \sqrt{\det C} = \frac{\rho_0}{\rho} = 1, \tag{6.48}
\]

where the first equality follows from (1.40) and (1.53). This condition places a restriction on \( C \), namely, the components of \( C \) are not all independent. In this particular case, the constraint (6.38) reads

\[
\lambda(C) = \det C - 1. \tag{6.49}
\]

Making use of the identity

\[
\frac{d}{dA} (\det A) = (\det A) A^{-T}, \tag{6.50}
\]

which is valid for all invertible second-order tensors \( A \), we derive

\[
\frac{d}{dC} (\det C - 1) = (\det C) C^{-1} = C^{-1}. \tag{6.51}
\]

Equation (6.44) then implies

\[
\Pi(t) = \alpha(t) C^{-1}(t). \tag{6.52}
\]

Instead of \( \alpha \) we write \(-p\) in order to indicate that \( p \) is a pressure. Equation (6.41) now reads

\[
T^{(2)}(t) = -p(t) C^{-1}(t) + G_{\tau \leq t} [C(\tau), \theta(\tau), \vec{G}_\theta(\tau)] . \tag{6.53}
\]

Complementary, the Cauchy stress tensor is expressed as

\[
t(t) = -p(t) I + R(t) \cdot F_{\tau \leq t} [C(\tau), \theta(\tau), \vec{G}_\theta(\tau)] \cdot R^T(t). \tag{6.54}
\]

This equation shows that for an incompressible simple material the stress is determined by the motion only to within a pressure \( p \).
6.7 Material symmetry

In this section, we will confine ourselves to the homogeneous material. We say that the material is homogeneous if the constitutive equations do not depend on the translations of the origin the reference configuration. In other words, there is at least one reference configuration in which a constitutive equation of a homogeneous material has the same form for all particles. It means that the explicit dependence of constitutive functionals on the position $\vec{X}$ disappears. For instance, equation (6.24) for a homogeneous material reduces to

$$t(\vec{X}, t) = \mathcal{F}_R[F(\vec{X}, \tau), \theta(\vec{X}, \tau), \bar{G}_\theta(\vec{X}, \tau)].$$

Note that the functional $\mathcal{F}$ is labelled by the subscript $R$ since $\mathcal{F}$ depends on the choice of reference configuration.

Material symmetry or material isotropy, if it exists, can be characterized by invariance properties of the constitutive equations with respect to a change of reference configuration. To make this statement more precise and to exploit its consequences, denote by $R$ and $\hat{R}$ two different reference configurations (see Figure 6.1) related by the one-to-one mapping

$$\hat{X} = \hat{\Lambda}(\vec{X}) \iff \vec{X} = \Lambda^{-1}(\hat{X}),$$

where $\vec{X}$ and $\hat{X}$ are the positions of the particle $X$ in the reference configurations $R$ and $\hat{R}$, respectively, $\vec{X} = X_K \vec{I}_K$ and $\hat{X} = \hat{X}_K \hat{I}_K$. Then the motion of the particle in the present
configuration \( \tilde{x} = \tilde{\chi}_R(\tilde{X}, t) \) takes the form \( \tilde{x} = \tilde{\chi}_R(\tilde{X}, t) \), and we obtain the identity
\[
\tilde{\chi}_R(\tilde{X}, t) = \tilde{\chi}_R(\tilde{\Lambda}(\tilde{X}), t),
\] which holds for all \( \tilde{X} \) and \( t \). Differentiating \( \tilde{x} = \tilde{\chi}_R(\tilde{X}, t) \) and \( \theta(\tilde{X}, t) \) with respect to \( \tilde{X} \) leads to the transformation rule for the deformation and temperature gradients:
\[
F_{kK} = \frac{\partial x_k}{\partial X_K} = \frac{\partial \chi_k}{\partial \Lambda_L} \frac{\partial \Lambda_L}{\partial X_K} = \hat{F}_{kl} P_{LK},
\]
\[
(\hat{G}_\theta)_K = \frac{\partial \theta}{\partial X_K} = \frac{\partial \theta}{\partial \Lambda_L} \frac{\partial \Lambda_L}{\partial X_K} = (\hat{G}_\theta)_L P_{LK},
\]
and in invariant notation,
\[
F(\tilde{X}, t) = \hat{F}(\tilde{x}, t) \cdot P(\tilde{X}),
\]
\[
\theta(\tilde{X}, t) = \hat{\theta}(\tilde{x}, t),
\]
\[
\hat{G}_\theta(\tilde{X}, t) = \hat{G}_\theta(\tilde{x}, t) \cdot P(\tilde{X}).
\] 
Here,
\[
\hat{F}(\tilde{X}, t) := (\text{Grad} \tilde{\chi}_R(\tilde{X}, t))^T
\]
denotes the transposed deformation gradient of the motion \( \tilde{x} = \tilde{\chi}_R(\tilde{X}, t) \), and
\[
P(\tilde{X}) := (\text{Grad} \tilde{\Lambda}(\tilde{X}))^T = P_{KL}(\tilde{X})(\hat{I}_K \otimes \hat{I}_L),
\]
is the transposed gradient of the transformation \( \tilde{\Lambda} \) that maps the configuration \( R \) onto the configuration \( \tilde{R} \). The deformation and temperature gradients therefore depend on the choice of configuration. Likewise, the Green deformation tensor \( C \), \( C = F^T \cdot F \), transforms according to the rule:
\[
C(\tilde{X}, t) = P^T(\tilde{X}) \cdot \hat{C}(\tilde{x}, t) \cdot P(\tilde{X}).
\] 
The constitutive equation (6.55) for the reference configuration \( \hat{R} \) is of the form
\[
t(\tilde{x}, t) = \mathcal{F}_R \left[ \hat{F}(\tilde{X}, \tau), \hat{\theta}(\tilde{X}, \tau), \hat{G}_\theta(\tilde{X}, \tau) \right].
\] 
Under the assumption that two different deformation and temperature histories are related by the same mapping as the associated reference configurations, the relation between the two different functionals \( \mathcal{F}_R \) and \( \mathcal{F}_{\tilde{R}} \) is given by the identity
\[
\mathcal{F}_R \left[ F(\tilde{X}, \tau), \theta(\tilde{X}, \tau), G_\theta(\tilde{X}, \tau) \right] = \mathcal{F}_{\tilde{R}} \left[ \tilde{F}(\tilde{X}, \tau), \tilde{\theta}(\tilde{X}, \tau), \tilde{G}_\theta(\tilde{X}, \tau) \right],
\] 
where (6.58) is implicitly considered. Thus, the arguments in configuration \( R \) can be expressed in terms of those in configuration \( \tilde{R} \):
\[
\mathcal{F}_R \left[ F(\tilde{X}, \tau) \cdot P(\tilde{X}), \theta(\tilde{X}, \tau), G_\theta(\tilde{X}, \tau) \cdot P(\tilde{X}) \right] = \mathcal{F}_{\tilde{R}} \left[ \tilde{F}(\tilde{X}, \tau), \tilde{\theta}(\tilde{X}, \tau), \tilde{G}_\theta(\tilde{X}, \tau) \right].
\]
Since (6.64) holds for any deformation and temperature histories, we obtain the identity
\[
\mathcal{F}_R \left[ F(\bar{X}, \tau) \cdot P(\bar{X}), \theta(\bar{X}, \tau), \tilde{G}_\theta(\bar{X}, \tau) \cdot P(\bar{X}) \right] = \mathcal{F}_R \left[ F(\bar{X}, \tau), \theta(\bar{X}, \tau), \tilde{G}_\theta(\bar{X}, \tau) \right] .
\] (6.65)

This identity can alternatively be arranged by expressing the arguments in configuration \( \tilde{R} \) in terms of those in configuration \( R \):
\[
\mathcal{F}_R \left[ F(\bar{X}, \tau), \theta(\bar{X}, \tau), \tilde{G}_\theta(\bar{X}, \tau) \right] = \mathcal{F}_R \left[ F(\bar{X}, \tau) \cdot P^{-1}(\bar{X}), \theta(\bar{X}, \tau), \tilde{G}_\theta(\bar{X}, \tau) \cdot P^{-1}(\bar{X}) \right] .
\] (6.66)

To introduce the concept of material symmetry, consider a particle which at time \( t = -\infty \) was in configuration \( R \) and subsequently suffered from certain histories of the deformation gradient \( F \), the temperature \( \theta \) and the temperature gradient \( \tilde{G}_\theta \). At the present time \( t \), the constitutive equation for the stress is
\[
t(\bar{x}, t) = \mathcal{F}_R \left[ F(\bar{x}, \tau), \theta(\bar{x}, \tau), \tilde{G}_\theta(\bar{x}, \tau) \right] .
\] (6.67)

Let the same particle but now in the configuration \( \tilde{R} \) is experienced the same histories of deformation gradient, temperature and temperature gradient as before. The resulting stress at present time \( t \) is
\[
t(\bar{x}, t) = \mathcal{F}_R \left[ F(\bar{x}, \tau), \theta(\bar{x}, \tau), \tilde{G}_\theta(\bar{x}, \tau) \right] .
\] (6.68)

Since \( \mathcal{F}_R \) and \( \mathcal{F}_\tilde{R} \) are not, in general, the same functionals, it follows that \( t \neq \tilde{t} \). However, it may happen that these values coincide, which expresses a certain symmetry of material. The condition for this case is
\[
\mathcal{F}_R \left[ F(\bar{x}, \tau), \theta(\bar{x}, \tau), \tilde{G}_\theta(\bar{x}, \tau) \right] = \mathcal{F}_R \left[ F(\bar{x}, \tau), \theta(\bar{x}, \tau), \tilde{G}_\theta(\bar{x}, \tau) \right] .
\] (6.69)

By expressing the right-hand side according to identity (6.65), we obtain
\[
\mathcal{F}_R \left[ F(\bar{x}, \tau) \cdot P(\bar{x}), \theta(\bar{x}, \tau), \tilde{G}_\theta(\bar{x}, \tau) \cdot P(\bar{x}) \right] = \mathcal{F}_R \left[ F(\bar{x}, \tau), \theta(\bar{x}, \tau), \tilde{G}_\theta(\bar{x}, \tau) \right] .
\] (6.70)

If this relation holds for all deformation and temperature histories, we say that the material at the particle \( \bar{x} \) is symmetric with respect to the transformation \( P : R \rightarrow \tilde{R} \).

It shows that any such \( P \) as in (6.70) corresponds to a static local deformation (at a given particle \( \bar{x} \)) from \( R \) to another reference configuration \( \tilde{R} \) such that any deformation and temperature histories lead to the same stress in either \( R \) or \( \tilde{R} \) (at a given particle \( \bar{x} \)). Hence \( P \) corresponds to a change of reference configuration which cannot be detected by any experiment at a given particle \( \bar{x} \). The matrices \( P \), for which (6.70) applies, are called the symmetry transformations. Often the constitutive functionals remain invariant under volume-preserving transformations, that is, the transformations with \( \det P = \pm 1 \). A transformation with this property is called unimodular. From now on, we will consider changes of reference configuration for which the associated deformation gradient \( P \) is unimodular. We point out that the unimodularity condition on the transformation of reference configuration could, in principle, be omitted. However, no materials are known that satisfy the symmetry condition (6.70) for a non-unimodular transformation.
The properties of material symmetry can be represented in terms of the symmetry group. The set of all unimodular transformations which leave the constitutive equation invariant with respect to $R$, that is, for which $(6.70)$ holds forms a group. ³ This group is called the symmetry group $g_R$ of the material at the particle $X$ whose place in the reference configuration $R$ is $\bar{X}$:

$$g_R := \{ H | \det H = \pm 1 \} \quad \text{and} \quad \mathcal{F}_R \left[ \begin{array}{c} F(\tau) \cdot H, \theta(\tau), \bar{G}_\theta(\tau) \cdot H \end{array} \right] = \mathcal{F}_R \left[ \begin{array}{c} F(\tau), \theta(\tau), \bar{G}_\theta(\tau) \end{array} \right] \quad \forall F(\tau), \forall \theta(\tau), \forall \bar{G}_\theta(\tau),$$

where we have dropped the place $\bar{X}$ from the notation as, for instance, in $F(\bar{X}, \tau) \equiv F(\tau)$. The operation defined on $g_R$ is the scalar product of tensors and the identity element is the identity tensor.

To show that $g_R$ is a group, let $H_1$ and $H_2$ be two elements of $g_R$. Then we can replace $F$ in $(6.71)$ with $F \cdot H_1$, $G_\theta$ with $G_\theta \cdot H_1$ and take $H = H_2$ to find

$$\mathcal{F}_R \left[ \begin{array}{c} F(\tau) \cdot H_1 \cdot H_2, \theta(\tau), \bar{G}_\theta(\tau) \cdot H_1 \cdot H_2 \end{array} \right] = \mathcal{F}_R \left[ \begin{array}{c} F(\tau) \cdot H_1, \theta(\tau), \bar{G}_\theta(\tau) \cdot H_1 \end{array} \right].$$

Since $H_1 \in g_R$, we find that

$$\mathcal{F}_R \left[ \begin{array}{c} F(\tau) \cdot H_1 \cdot H_2, \theta(\tau), \bar{G}_\theta(\tau) \cdot H_1 \cdot H_2 \end{array} \right] = \mathcal{F}_R \left[ \begin{array}{c} F(\tau), \theta(\tau), \bar{G}_\theta(\tau) \end{array} \right].$$

This shows that the scalar product $H_1 \cdot H_2 \in g_R$. Furthermore, the identity tensor $H = I$ clearly satisfies $(6.71)$. Finally, if $H \in g_R$ and $F$ is invertible, then $F \cdot H^{-1}$ is an invertible tensor. Hence $(6.71)$ must hold with $F$ replaced by $F \cdot H^{-1}$ (and $G_\theta$ with $G_\theta \cdot H^{-1}$):

$$\mathcal{F}_R \left[ \begin{array}{c} F(\tau) \cdot H^{-1} \cdot H, \theta(\tau), \bar{G}_\theta(\tau) \cdot H^{-1} \cdot H \end{array} \right] = \mathcal{F}_R \left[ \begin{array}{c} F(\tau) \cdot H^{-1}, \theta(\tau), \bar{G}_\theta(\tau) \cdot H^{-1} \end{array} \right].$$

Hence we find that

$$\mathcal{F}_R \left[ \begin{array}{c} F(\tau), \theta(\tau), \bar{G}_\theta(\tau) \end{array} \right] = \mathcal{F}_R \left[ \begin{array}{c} F(\tau) \cdot H^{-1}, \theta(\tau), \bar{G}_\theta(\tau) \cdot H^{-1} \end{array} \right],$$

which shows that $H^{-1} \in g_R$. We have proved that $g_R$ has the structure of a group. Note that $(6.72)$ is an equivalent condition for the material symmetry.

### 6.8 Material symmetry of reduced-form functionals

We now employ the definition of the symmetry group to find a symmetry criterion for the functionals whose forms are reduced by applying the polar decomposition of the deformation gradient. As an example, let us consider the constitutive equation $(6.30)$ for the Cauchy stress tensor. For

³A group is a set of abstract elements $A, B, C, \cdots$, with a defined operation $A \cdot B$ (as, for instance, ‘multiplication’) such that: (1) the product $A \cdot B$ is defined for all elements $A$ and $B$ of the set, (2) this product $A \cdot B$ is itself an element of the set for all $A$ and $B$ (the set is closed under the operation), (3) the set contains an identity element $I$ such that $I \cdot A = A = A \cdot I$ for all $A$, (4) every element has an inverse $A^{-1}$ in the set such that $A \cdot A^{-1} = A^{-1} \cdot A = I$. For
a homogeneous material, this constitutive equation in the reference configurations $R$ and $\hat{R}$ has the form

$$t(\bar{X}, t) = F(\bar{X}, t) \cdot \tilde{F}_R \left[ C(\bar{X}, \tau), \theta(\bar{X}, \tau), \tilde{G}_\theta(\bar{X}, \tau) \right] \cdot F^T(\bar{X}, t),$$

$$t(\bar{X}, t) = \tilde{F}(\bar{X}, t) \cdot \tilde{F}_\tilde{R} \left[ C(\bar{X}, \tau), \theta(\bar{X}, \tau), \tilde{G}_\tilde{\theta}(\bar{X}, \tau) \right] \cdot F^T(\bar{X}, t),$$

(6.73)

where the subscripts $R$ and $\tilde{R}$ refer to the underlying reference configuration. With the help of the transformation relation (6.58) for the deformation gradient, the temperature and the temperature gradient, and (6.61) for the Green deformation tensor, the functional $\tilde{F}_R$ satisfies the identity analogous to (6.66):

$$P(\bar{X}) \cdot \tilde{F}_R \left[ C(\bar{X}, \tau), \theta(\bar{X}, \tau), \tilde{G}_\theta(\bar{X}, \tau) \right] \cdot P^T(\bar{X})$$

$$= \tilde{F}_R \left[ P^{-T}(\bar{X}) \cdot C(\bar{X}, \tau) \cdot P^{-1}(\bar{X}), \theta(\bar{X}, \tau), \tilde{G}_\theta(\bar{X}, \tau) \cdot P^{-1}(\bar{X}) \right].$$

(6.74)

Since $\tilde{F}_R$ and $\tilde{F}_\tilde{R}$ are not, in general, the same functionals, the functional $\tilde{F}_R$ on the right-hand side of (6.74) cannot be replaced by the functional $\tilde{F}_\tilde{R}$. However, it may happen, that $\tilde{F}_R = \tilde{F}_\tilde{R}$, which expresses a certain symmetry of material. The condition for this case is

$$P(\bar{X}) \cdot \tilde{F}_R \left[ C(\bar{X}, \tau), \theta(\bar{X}, \tau), \tilde{G}_\theta(\bar{X}, \tau) \right] \cdot P^T(\bar{X})$$

$$= \tilde{F}_R \left[ P^{-T}(\bar{X}) \cdot C(\bar{X}, \tau) \cdot P^{-1}(\bar{X}), \theta(\bar{X}, \tau), \tilde{G}_\theta(\bar{X}, \tau) \cdot P^{-1}(\bar{X}) \right].$$

(6.75)

We say that a material at the particle $\mathcal{X}$ (whose place in the reference configuration is $\bar{X}$) is symmetric with respect to the transformation $P : R \to \hat{R}$ if the last relation is valid for all deformation and temperature histories. The symmetry group $g_R$ of a material characterized by the constitutive functional $\tilde{F}_R$ is defined as

$$g_R := \left\{ H \mid \det H = \pm 1 \right\}$$

and

$$H \cdot \tilde{F}_R \left[ C(\tau), \theta(\tau), \tilde{G}_\theta(\tau) \right] \cdot H^T = \tilde{F}_R \left[ H^{-T} \cdot C(\tau) \cdot H^{-1}, \theta(\tau), \tilde{G}_\theta(\tau) \cdot H^{-1} \right]$$

(6.76)

for $\forall C(\tau), \forall \theta(\tau), \forall \tilde{G}_\theta(\tau)$,

where we have dropped the place $\bar{X}$ from the notation as, for instance, in $C(\bar{X}, \tau) \equiv C(\tau)$.

By an analogous way, it can be shown that the condition of material symmetry for the materially objective constitutive functional $G_R$ in constitutive equation (6.34) is similar to that of $\tilde{F}_R$:

$$H \cdot G_R \left[ C(\tau), \theta(\tau), \tilde{G}_\theta(\tau) \right] \cdot H^T = \tilde{G}_R \left[ H^{-T} \cdot C(\tau) \cdot H^{-1}, \theta(\tau), \tilde{G}_\theta(\tau) \cdot H^{-1} \right].$$

(6.77)

6.9 Noll’s rule

The symmetry group $g_R$ generally depends upon the choice of reference configuration, since the symmetries a body enjoys with respect to one configuration generally differ from those it enjoys
with respect to another. However, the symmetry groups \( g_R \) and \( g_{\hat{R}} \) (for the same particle) relative to two different reference configurations \( R \) and \( \hat{R} \) are related by Noll’s rule:

\[
g_{\hat{R}} = P \cdot g_R \cdot P^{-1},
\]

where \( P \) is the fixed local deformation tensor for the deformation from \( R \) to \( \hat{R} \) given by (6.60). Noll’s rule means that every tensor \( \hat{H} \in g_{\hat{R}} \) must be of the form \( \hat{H} = P \cdot H \cdot P^{-1} \) for some tensor \( H \in g_R \), and conversely every tensor \( H \in g_R \) is of the form \( H = P^{-1} \cdot \hat{H} \cdot P \) for some \( \hat{H} \in g_{\hat{R}} \). Note that \( P \) not necessarily be unimodular, because \( P \) represents a change of reference configuration, not a member of symmetry group. Noll’s rule shows that if \( g_R \) is known for one configuration, it is known for all. That is, the symmetries of a material in any one configuration determine its symmetries in every other.

To prove Noll’s rule, let \( H \in g_R \). Then (6.71) holds and it is permissible to replace \( F \) by \( F \cdot P \) and \( \vec{G}_\theta \) by \( \vec{G}_\theta \cdot P \):

\[
\mathcal{F}_R \left[ F(\tau) \cdot P \cdot H, \theta(\tau), \vec{G}_\theta(\tau) \cdot P \cdot H \right] = \mathcal{F}_R \left[ F(\tau) \cdot P, \theta(\tau), \vec{G}_\theta(\tau) \cdot P \right].
\]

By applying identity (6.66) to the both side of this equation, we obtain

\[
\mathcal{F}_R \left[ F(\tau) \cdot P \cdot H \cdot P^{-1}, \theta(\tau), \vec{G}_\theta(\tau) \cdot P \cdot H \cdot P^{-1} \right] = \mathcal{F}_R \left[ F(\tau), \theta(\tau), \vec{G}_\theta(\tau) \right],
\]

which holds for all deformation and temperature histories. Thus, if \( H \in g_R \), then \( P \cdot H \cdot P^{-1} \in g_{\hat{R}} \).

The argument can be reversed which completes the proof of Noll’s rule.

By definition, the symmetry group is a subgroup of the group of all unimodular transformations,

\[
g_R \subseteq \text{unim}. \quad (6.79)
\]

The ‘size’ of a specific symmetry group is the measure for the amount of material symmetry. The group of unimodular transformations contains the group of orthogonal transformations (as, for instance, rotations or reflections) but also the group of non-orthogonal volume-preserving transformations (as, for instance, torsions), i.e.,

\[
\text{orth.} \subset \text{unim}. \quad (6.80)
\]

The smallest symmetry group \( \{I, -I\} \subset g_R \) corresponds to a material that has no (nontrivial) symmetries. Such a material is called triclinic. Obviously,

\[
P \cdot \{I, -I\} \cdot P^{-1} = \{I, -I\}.
\]

Hence a triclinic material has no symmetries relative to any reference configuration: no transformation can bring this material into a configuration that has a nontrivial symmetry.

The largest symmetry group is the group of all unimodular transformations. Noll’s rule shows that if \( P \) is an arbitrary invertible tensor, then \( P \cdot H \cdot P^{-1} \) is unimodular for all unimodular \( H \), and if \( \overline{H} \) is any unimodular tensor, the tensor \( P^{-1} \cdot \overline{H} \cdot P \) is a unimodular tensor, and \( \overline{H} = P \cdot H \cdot P^{-1} \). Therefore,

\[
P \cdot \text{unim.} \cdot P^{-1} = \text{unim.} \quad (6.82)
\]
Hence, the maximum symmetry of such a material cannot be destroyed by deformation.

6.10 Classification of the symmetry properties

6.10.1 Isotropic materials

We will now classify material according to the symmetry group it belongs to. We say that a material is isotropic if it possesses a configuration \( R \), in which the symmetry condition \((6.71) \) or \((6.76) \) applies at least for all orthogonal transformations. In other words, a material is isotropic if there exists a reference configuration \( R \) such that

\[
g_R \supseteq \text{orth.}
\]

where \text{orth.} denotes the group of all orthogonal transformation. For an isotropic material, \((6.79) \) and \((6.83) \) can be combined to give

\[
\text{orth.} \subseteq g_R \subseteq \text{unim}.
\]

All groups of isotropic bodies are thus bounded by the orthogonal and unimodular groups. A question arises of how many groups bounded by \text{orth.} and \text{unim.} may exist. The group theory states that the orthogonal group is a maximal subgroup of the unimodular group. Consequently, there are only two groups satisfying \((6.84) \), either orthogonal or unimodular; no other group exists between them. Hence, there are two kinds of isotropic materials, either those with \( g_R = \text{orth.} \) or those with \( g_R = \text{unim} \). All other materials are anisotropic and possess a lower degree of symmetry.

The concept of the symmetry group can be used to define solids and fluids according to Noll.

6.10.2 Fluids

Fluids have the property that can be poured from one container to another and, after some time, no evidence of the previous circumstances remains. Such a change of container can be thought as a change of reference configuration, so that for fluids all reference configurations with the same density are indistinguishable. In other words, every configuration, also the present configuration, can be thought as a reference configuration. According to Noll, a simple material is a fluid if its symmetry group has a maximal symmetry, being identical with the set of all unimodular transformations,

\[
g_R = \text{unim}.
\]

Moreover, Noll’s rule implies that the condition \( g_R = \text{unim} \) holds for every configuration if it holds for any one reference configuration \( R \), so a fluid has maximal symmetry relative to every reference configuration. A fluid is thus a non-solid material with no preferred configurations. In terms of symmetry groups, \( g_R = g_R (= \text{unim}) \). Moreover, a fluid is isotropic, because of \((6.80) \).

The simple fluid and the simple solid do not exhaust the possible types of simple materials. There is, for instance, a simple material, called liquid crystal which is neither a simple fluid nor a simple solid.

6.10.3 Solids

Solids have the property that any change of shape (as represented by a non-orthogonal transformation of the reference configuration) brings the material into a new reference configuration.
from which its response is different. Hence, according to Noll, a simple material is called a solid if there is a reference configuration such that every element of the symmetry group is a rotation,

$$g_R \subseteq \text{orth.} \quad (6.86)$$

A solid whose symmetry group is equal to the full orthogonal group is called isotropic,

$$g_R = \text{orth.} \quad (6.87)$$

If the symmetry group of a solid is smaller than the full orthogonal group, the solid is called anisotropic,

$$g_R \subset \text{orth.} \quad (6.88)$$

There is only a finite number of symmetry group with this property, and it forms the 32 crystal classes.

### 6.11 Constitutive equation for isotropic materials

Every material belonging to a certain symmetry group $g_R$ must also be objective. Thus, the objectivity condition (6.25) and the condition of material symmetry (6.72) must be satisfied simultaneously. Combining them, the material functional must satisfied the following condition

$$O(t) \cdot F_{\tau \leq t} \left[ F(\tau), \theta(\tau), \bar{G}_{\theta}(\tau) \right] \cdot O^T(t) = F_{\tau \leq t} \left[ O(\tau) \cdot F(\tau) \cdot H^{-1}, \theta(\tau), \bar{G}_{\theta}(\tau) \cdot H^{-1} \right] \quad (6.89)$$

for all orthogonal transformation $O(t)$ of the reference frame and all symmetry transformation $H \in g_R$. Note that the subscript $R$ at the functional $F$ is omitted since we will not consider more than one reference configuration in the following text. For an isotropic material, the set of unimodular transformations is equal to the set of full orthogonal transformations, $H = Q \in \text{orth}$. In contrast to $O(t)$, the orthogonal tensor $Q$ is time independent. Restricting the last transformation to constant-in-time element $Q$, (6.89) becomes

$$Q \cdot F_{\tau \leq t} \left[ F(\tau), \theta(\tau), \bar{G}_{\theta}(\tau) \right] \cdot Q^T = F_{\tau \leq t} \left[ Q \cdot F(\tau) \cdot Q^T, \theta(\tau), Q \cdot \bar{G}_{\theta}(\tau) \right], \quad (6.90)$$

where we have identified $Q \cdot \bar{G}_{\theta}(\tau) \equiv \bar{G}_{\theta}(\tau) \cdot Q^T$. Similar constraints can be derived for the constitutive functionals $Q$ and $\mathcal{E}$ of the heat flux and the internal energy, respectively:

$$Q \cdot F_{\tau \leq t} \left[ F(\tau), \theta(\tau), \bar{G}_{\theta}(\tau) \right] = Q \cdot F_{\tau \leq t} \left[ Q \cdot F(\tau) \cdot Q^T, \theta(\tau), Q \cdot \bar{G}_{\theta}(\tau) \right], \quad (6.91)$$

Functionals which satisfy constraints (6.89) and (6.91) are called tensorial, vectorial and scalar isotropic functionals with respect to orthogonal transformations. All these functionals represent constitutive equations for an isotropic body. Note that the conditions (6.89)–(6.91) are necessary for the material objectivity of isotropic functionals, but not necessarily sufficient, since the principle of the material objectivity is satisfied for a constant, time-independent $Q \in \text{orth}$, but not for a general $O(t) \in \text{orth}$. In many cases, the functionals are also materially objective for all $O(t) \in \text{orth}$, but this must be examined in every individual case.
6.12 Current configuration as reference

So far, we have employed a reference configuration fixed in time, but we can also use a reference configuration varying in time. Thus one motion may be described in terms of any other. The only time-variable reference configuration useful in this way is the present configuration. If we take it as reference, we describe past events as they seem to an observer fixed to the particle $X$ now at the place $\vec{x}$. The corresponding description is called relative and it has been introduced in section 1.2.

To see how such the relative description is constructed, consider the configurations of body $B$ at the two times $\tau$ and $t$:

\[
\begin{align*}
\vec{\xi} &= \vec{\chi}(\vec{X}, \tau), \\
\vec{x} &= \vec{\chi}(\vec{X}, t),
\end{align*}
\]

(6.92)

that is, $\xi$ is the place occupied at time $\tau$ by the particle that occupies $\vec{x}$ at time $t$. Since the function $\vec{\chi}$ is invertible, that is, $\vec{X} = \vec{\chi}^{-1}(\vec{x}, t)$,

(6.93)

we have either

\[
\vec{\xi} = \vec{\chi}(\vec{\chi}^{-1}(\vec{x}, t), \tau) =: \vec{\chi}_t(\vec{x}, \tau)
\]

(6.94)

where the function $\vec{\chi}_t(\vec{x}, \tau)$ is called the relative motion function. It describes the deformation of the new configuration $\kappa_t$ of the body $B$ relative to the present configuration $\kappa_t$, which is considered as reference. The subscript $t$ at the function $\vec{\chi}$ is used to recall this fact. The relative description, given by the mapping (6.94), is actually a special case of the referential description, differing from the Lagrangian description in that the reference position is now denoted by $\vec{x}$ at time $t$ instead of $\vec{X}$ at time $t = 0$. The variable time $\tau$, being the time when the particle occupied the position $\xi$, is now considered as an independent variable instead of the time $t$ in the Lagrangian description.

The relative deformation gradient $F_t$ is the gradient of the relative motion function:

\[
F_t(\vec{x}, \tau) := (\text{grad} \, \vec{\chi}_t(\vec{x}, \tau))^T, \quad (F_t)_{kj} := \frac{\partial (\vec{\chi}_t)_k}{\partial x_l}.
\]

(6.95)

It can be expressed in terms of the (absolute) deformation gradient $F$. Differentiating the following identity for the motion function,

\[
\dot{\vec{\xi}} = \vec{\chi}(\dot{\vec{X}}, \tau) = \vec{\chi}_t(\vec{x}, \tau) = \vec{\chi}_t(\vec{\chi}(\vec{X}, t), \tau),
\]

with respect to $X_K$ and using the chain rule of differentiation, we get

\[
\frac{\partial \chi_k(\vec{X}, \tau)}{\partial X_K} = \frac{\partial (\vec{\chi}_t)_k(\vec{x}, \tau)}{\partial x_l} \frac{\partial \chi_l(\vec{X}, t)}{\partial X_K}
\]

or

\[
(F)_{kK}(\vec{X}, \tau) = (F_t)_{kl}(\vec{x}, \tau)(F)_{lK}(\vec{X}, t),
\]

which represents the tensor equation

\[
F(\vec{X}, \tau) = F_t(\vec{x}, \tau) \cdot F(\vec{X}, t), \quad \text{or} \quad F_t(\vec{x}, \tau) = F(\vec{X}, \tau) \cdot F^{-1}(\vec{x}, t).
\]

(6.96)

In particular,

\[
F_t(\vec{x}, t) = I.
\]

(6.97)
The Green deformation tensor \( C = F^T \cdot F \) may also be developed in terms of the relative deformation gradient:

\[
C(\vec{X}, \tau) = F^T(\vec{X}, \tau) \cdot F(\vec{X}, \tau) = F^T(\vec{x}, t) \cdot F^{-1}(\vec{x}, \tau) \cdot F(\vec{x}, t) .
\]

Defining the relative Green deformation tensor by

\[
\begin{align*}
\mathbf{c}_t(\vec{x}, \tau) &:= F^T_t(\vec{x}, \tau) \cdot F_t(\vec{x}, \tau), \\
\mathbf{G}_\theta(\tau) &:= \mathbf{g}_\theta(\vec{x}, \tau) = \partial_\theta(\vec{x}, \tau) = \partial_\theta(\vec{X}(\tau), \tau) \\
&= \mathbf{R}^T(\vec{X}(\tau)) \cdot \mathbf{G}_\theta(\vec{X}(\tau)),
\end{align*}
\]

(6.98)

we have

\[
C(\vec{X}, \tau) = F^T(\vec{x}, t) \cdot \mathbf{c}_t(\vec{x}, \tau) \cdot F(\vec{x}, t) .
\]

(6.99)

6.13 Isotropic constitutive functionals in relative representation

Let now the motion be considered in relative representation (6.94). Thus, the present configuration at the current time \( t \) serves as reference, while the configuration at past time \( \tau, (\tau \leq t) \), is taken as present. The relative deformation gradient \( F_t(\vec{x}, \tau) \) can be expressed in terms of the absolute deformation gradient according to (6.96). Similarly, the relative temperature gradient, \( \mathbf{g}_\theta := \mathbf{g}_\theta(\vec{x}, \tau) = \partial_\theta(\vec{x}, \tau) \), can be expressed in terms of the absolute temperature gradient, \( \mathbf{G}_\theta := \mathbf{Grad}_\theta(\vec{x}, \tau) \), as

\[
\begin{align*}
\mathbf{g}_\theta(\vec{x}, \tau) &= \frac{\partial(\vec{x}(\tau))}{\partial x_k} \cdot \frac{\partial \theta(\vec{X}(\tau))}{\partial X_k} \cdot \frac{\partial \theta(\vec{x}(\tau))}{\partial x_k} \\
&= \mathbf{G}_\theta(\vec{x}(\tau)) \cdot \mathbf{F}^{-1}(\vec{x}, t) = \mathbf{F}^{-T}(\vec{x}, t) \cdot \mathbf{G}_\theta(\vec{x}(\tau)).
\end{align*}
\]

To abbreviate notations, \( (\vec{X}, t) \) and \( (\vec{x}, t) \) will be dropped from notations and \( (\vec{X}, \tau) \) and \( (\vec{x}, \tau) \) will be shortened as \( (\tau) \):

\[
\mathbf{G}_\theta(\tau) = \mathbf{F}^T \cdot \mathbf{g}_\theta(\tau). 
\]

(6.100)

With the help of the polar decomposition \( \mathbf{F} = \mathbf{R} \cdot \mathbf{U} \), (6.99) and (6.100) can also be written in the form

\[
\begin{align*}
C(\tau) &= \mathbf{U} \cdot \mathbf{R}^T \cdot \mathbf{c}_t(\tau) \cdot \mathbf{R} \cdot \mathbf{U}, \\
\mathbf{G}_\theta(\tau) &= \mathbf{U} \cdot \mathbf{R}^T \cdot \mathbf{g}_\theta(\tau).
\end{align*}
\]

(6.101)

We consider the reduced form (6.29) of the constitutive equation for the Cauchy stress tensor:

\[
\mathbf{t} = \mathbf{R} \cdot \mathbf{F} \left[ C(\tau), \theta(\tau), \mathbf{G}_\theta(\tau) \right] \cdot \mathbf{R}^T.
\]

(6.102)

Making use of (6.101) together with \( \mathbf{C} = \mathbf{U}^2 \), we obtain

\[
\mathbf{t} = \mathbf{R} \cdot \mathbf{F} \left[ \sqrt{\mathbf{C}} \cdot \mathbf{R}^T \cdot \mathbf{c}_t(\tau) \cdot \mathbf{R} \cdot \sqrt{\mathbf{C}}, \theta(\tau), \sqrt{\mathbf{C}} \cdot \mathbf{R}^T \cdot \mathbf{g}_\theta(\tau) \right] \cdot \mathbf{R}^T.
\]

(6.103)

The information contained in the last constitutive equation can alternatively be expressed as

\[
\mathbf{t} = \mathbf{R} \cdot \mathbf{F} \left[ \mathbf{R}^T \cdot \mathbf{c}_t(\tau) \cdot \mathbf{R}, \theta(\tau), \mathbf{R}^T \cdot \mathbf{g}_\theta(\tau); \mathbf{C} \right] \cdot \mathbf{R}^T,
\]

(6.104)

where the present strain \( \mathbf{C}(\tau) \) appears in the functional \( \mathbf{F} \) as a parameter. This shows that it is not possible to express the effect of deformation history on the stress entirely by measuring
deformation with respect to the present configuration. While the effect of all the past history is accounted for, a fixed reference configuration is required, in general, to allow for the effect of the deformation at the present instant, as indicated by the appearance of $C(t)$ as a parameter in (6.104).

We are searching for a form of functional $F$ for isotropic materials only. In this case, the functional $F$ satisfies the isotropy relation analogous to (6.76) (show it):

$$F \cdot \frac{\partial \mathbf{F}}{\partial \mathbf{r}} \left[ R^T \cdot \mathbf{c}_l(\tau) \cdot R, \theta(\tau), R^T \cdot \tilde{g}_b(\tau); C \right] \cdot Q^T = 0$$

(6.105)

where $Q$ is an orthogonal tensor. Since $\mathbf{X}$ and $t$ are fixed in the relative description of motion, we may choose $Q(\mathbf{X}) = R(\mathbf{X}, t)$, or more explicitly, $Q_{KL}(\mathbf{X}) = Q_{KL}(\mathbf{X}) = R_{kl}(\mathbf{X}, t)$:

$$F \cdot \frac{\partial \mathbf{F}}{\partial \mathbf{r}} \left[ c_l(\tau), \theta(\tau), \tilde{g}_b(\tau); b \right] = R \cdot F \cdot \frac{\partial \mathbf{F}}{\partial \mathbf{r}} \left[ R^T \cdot c_l(\tau) \cdot R, \theta(\tau), R^T \cdot \tilde{g}_b(\tau); C \right] \cdot R^T,$$

(6.106)

where $b$ is the Finger deformation tensor, $b = F \cdot F^T = R \cdot C \cdot R^T$. Equations (6.104) and (6.106) can be combined to give

$$t = F \cdot \frac{\partial \mathbf{F}}{\partial \mathbf{r}} \left[ c_l(\tau), \theta(\tau), \tilde{g}_b(\tau); b \right].$$

(6.107)

This is a general form of the constitutive equation for a simple homogeneous isotropic materials expressed in relative representation. It remains to show that this constitutive equation is invariant to any orthogonal transformation $Q$ of reference configuration $R$, not only to our special choice $Q = R(\mathbf{X}, t)$. Since none of the arguments in (6.107) depends on the reference configuration $R$, this requirement is satisfied trivially. For instance, the Finger deformation tensor does not change by the orthogonal transformation of the reference configuration $R$ onto $\hat{R}$:

$$\hat{b} = \hat{F} \cdot \hat{F}^T = (F \cdot Q^{-1}) \cdot (Q^{-T} \cdot F^T) = F \cdot Q^{-1} \cdot Q \cdot F^T = F \cdot F^T = b.$$

By construction, the constitutive functional $F$ is not automatically materially objective. The material objectivity (6.16) of the Cauchy stress tensor implies that the functional $F$ must transform under a rigid motion of spatial frame as

$$F \cdot \frac{\partial \mathbf{F}}{\partial \mathbf{r}} \left[ c_l^*(\tau), \theta^*(\tau), \tilde{g}_b^*(\tau); b^* \right] = O(t) \cdot F \cdot \frac{\partial \mathbf{F}}{\partial \mathbf{r}} \left[ c_l(\tau), \theta(\tau), \tilde{g}_b(\tau); b \right] \cdot O^T(t).$$

(6.108)

The transformation rules for the relative deformation tensor, the relative Green deformation tensor, the relative temperature gradient and the Finger deformation tensor are

$$F^*_l(\tau) = O(\tau) \cdot F_l(\tau) \cdot O^T(t),$$

$$c^*_l(\tau) = O(t) \cdot c_l(\tau) \cdot O^T(t),$$

$$\tilde{g}_b^*(\tau) = O(t) \cdot \tilde{g}_b(\tau),$$

(6.109)

$$b^*(t) = O(t) \cdot b(t) \cdot O^T(t).$$

The proof is immediate:

$$F^*_l(\tau) = F^*(\tau) \cdot [F^*(t)]^{-1} = O(\tau) \cdot F(\tau) \cdot F^{-1}(t) \cdot O^{-1}(t) = O(\tau) \cdot F_l(\tau) \cdot O^T(t),$$

$$c^*_l(\tau) = [F^*(t)]^{-T} \cdot C^*(\tau) \cdot [F^*(t)]^{-1} = O(t) \cdot F^{-T}(t) \cdot C(t) \cdot F^{-1}(t) \cdot O^{-1}(t) = O(t) \cdot c_l(\tau) \cdot O^T(t),$$

$$\tilde{g}_b^*(\tau) = [F^*(t)]^{-T} \cdot \tilde{G}_b^*(\tau) = O(t) \cdot F^{-T}(t) \cdot \tilde{G}_b(\tau) = O(t) \cdot \tilde{g}_b(\tau).$$
By this and realizing that the temperature $\theta(\tau)$ as a scalar quantity is invariant to any transformation of spatial frame, the constraint (6.108) reduces to
\[
\mathcal{F}_{\tau \leq t} \left[ O(t) \cdot c_l(\tau) \cdot O^T(t), \theta(\tau), O(t) \cdot g_\theta(\tau); O(t) \cdot b \cdot O^T(t) \right] = O(t) \cdot \mathcal{F}_{\tau \leq t} \left[ c_l(\tau), \theta(\tau), g_\theta(\tau); b \right] \cdot O^T(t) .
\] (6.110)

This shows that the functional $\mathcal{F}$ must be a spatially isotropic functional of the variables $c_l(\tau)$, $\theta(\tau)$, $g_\theta(\tau)$ and a spatially isotropic function of $b$ for $\mathcal{F}$ to be materially objective.  

An analogous arrangement can be carried out for the constitutive equation (6.31) for the heat flux and the internal energy:
\[
\vec{q} = \mathcal{F}_{\tau \leq t} \left[ c_l(\tau), \theta(\tau), g_\theta(\tau); b \right], \\
\varepsilon = \mathcal{E}_{\tau \leq t} \left[ c_l(\tau), \theta(\tau), g_\theta(\tau); b \right] .
\] (6.111)

Note that the constitutive equations in relative representation will be used to describe material properties of a fluid, while for a solid we will employ the constitutive equation (6.29) in referential representation.

### 6.14 A general constitutive equation of a fluid

A fluid is characterized through the largest symmetry group, being identical with the set of all unimodular transformations, $g_R = \text{unim}$. Mathematically, the condition for the material symmetry of functional $\mathcal{F}$ in (6.107) under unimodular transformation $H$ is expressed as
\[
\mathcal{F}_{\tau \leq t} \left[ c_l(\tau), \theta(\tau), g_\theta(\tau); F \cdot F^T \right] = \mathcal{F}_{\tau \leq t} \left[ c_l(\tau), \theta(\tau), g_\theta(\tau); F \cdot H \cdot H^T \cdot F^T \right] ,
\] (6.112)
since the Finger deformation tensor $b = F \cdot F^T$ transforms under unimodular transformation $H$ as $F \cdot H \cdot H^T \cdot F^T$. This general characterization of a fluid must also hold for the special unimodular transformation
\[
H = \left( \det F \right)^{1/3} F^{-1} = \left( \frac{\rho_0}{\rho} \right)^{1/3} F^{-1} ,
\] (6.113)
where $\rho(\vec{x}, t)$ and $\rho_0(\vec{X})$ is the mass density of a body in the present and reference configuration, respectively. The transformation (6.113) is indeed unimodular since
\[
\det H = \left[ \left( \det F \right)^{1/3} \right]^3 \det F^{-1} = 1 .
\]

For such $H$, $F \cdot H \cdot H^T \cdot F^T = \left( \det F \right)^{2/3} I = (\rho_0/\rho)^{2/3} I$, and the constitutive equation (6.107) for an isotropic body can be rewritten with new denotation:
\[
t = \mathcal{F}_{\tau \leq t} \left[ c_l(\tau), \theta(\tau), g_\theta(\tau); \rho \right] .
\] (6.114)

\footnote{Note that this is spatial isotropy, not material isotropy discussed in previous sections.}
This is a general form of the constitutive equation for a simple fluid in relative representation. It shows that the Cauchy stress tensor for fluid depends on the relative deformation history, the temperature and temperature gradient histories, and on the present mass density as a parameter. Since none of $c_t(\vec{x}, \tau)$, $\theta(\vec{x}, \tau)$, $\vec{g}_\theta(\vec{x}, \tau)$ and $\theta(\vec{x}, t)$ does depend on the reference configuration, constitutive equation (6.114) is invariant to any transformation of the reference configuration. Hence, it satisfies trivially the requirement that $g_R = \text{unim}$. In other words, a fluid is a body for which every configuration that leaves the density unchanged can be taken as the reference configuration.

6.15 The principle of bounded memory

This principle states that deformation and temperature events at distant past from the present have a very small influence on the present behavior of material. In other words, the memory of past motions and temperatures of any material point decays rapidly with time. This principle is the counterpart of the principle of smooth neighborhood in the time domain. No unique mathematical formulation can be made of this principle. The following limited interpretation suffices for our purpose.

To express the boundedness of a memory, let $h(\vec{X}, \tau)$ be a function in the argument of constitutive functional. Suppose that $h(\vec{X}, \tau)$ is an analytical function such that it possesses continuous partial derivatives with respect to $\tau$ at $\tau = t$. For small $t - \tau$, it can be approximated by the truncated Taylor series expansion at $\tau = t$:

$$h(\vec{X}, \tau) = \sum_{n=0}^{N} \frac{(-1)^n}{n!} \frac{\partial^n h(\vec{X}, \tau)}{\partial \tau^n} \bigg|_{\tau=t} (t - \tau)^n. \quad (6.115)$$

If a constitutive functional is sufficiently smooth so that the dependence on $h(\vec{X}, \tau)$ can be replaced by the list of functions

$$^{(n)}h(\vec{X}, t) := \frac{\partial^n h(\vec{X}, \tau)}{\partial \tau^n} \bigg|_{\tau=t} \quad \text{for} \quad n = 0, 1, \ldots, N, \quad (6.116)$$

that is, $^{(n)}h$ is the $n$th material time derivative of $h$, $^{(n)}h = D^n h / D\tau^n$, we say that the material is of the rate type of degree $N$ with respect to the variable $h$. If the material is of the rate type in all its variables, the constitutive equation (6.34) for a homogeneous solid can be approximated by

$$T^{(2)} = T^{(2)}(C, \dot{C}, \ldots, ^{(N_C)}C, \theta, \dot{\theta}, \ldots, ^{(N_\theta)}\theta, \vec{G}_\theta, \hat{G}_\theta, \ldots, ^{(N_G)}\vec{G}_\theta). \quad (6.117)$$

The number of time derivatives for each of the gradients depends on the strength of the memory of these gradients. We note that $T^{(2)}$ is no longer functional. It is a tensor-valued function of the arguments listed. They involve time rates of the deformation gradient up to the order $N_C$ and temperature and its gradient up to order $N_\theta$ and $N_G$, respectively. The degrees $N_C$, $N_\theta$ and $N_G$ need not be the same.  

\[5\] The functions in the argument of constitutive functionals may not be so smooth as to admit truncated Taylor series expansion. Nevertheless, the constitutive functionals may be such as to smooth out past discontinuities in these argument functions and/or their derivatives. The principle of bounded memory, in this context also called
The general constitutive equation for a fluid is given by (6.114). We again assume that the relative Green deformation tensor \( c_t(\tau) \) can be approximated by the truncated Taylor series expansion of the form (6.115):

\[
c_t(\tau) = a_0(t) - a_1(t)(\tau - t) + a_2(t)(\tau - t)^2/2 + \ldots .
\]  

Using (6.99), the expansion coefficient

\[
a_n(\vec{x}, t) := \left. \frac{\partial^n c_t(\vec{x}, \tau)}{\partial \tau^n} \right|_{\tau=t}
\]

can be shown to be the Rivlin-Ericksen tensor of order \( n \) defined by (2.29). It is an objective tensor:

\[
a^*_n(\vec{x}, t) = O(t) \cdot a_n(\vec{x}, t) \cdot O^T(t).
\]

To show this, we use (6.109):

\[
a^*_n(\vec{x}, t) = \left. \frac{\partial^n c_t^*(\vec{x}, \tau)}{\partial \tau^n} \right|_{\tau=t} = \left. \frac{\partial^n}{\partial \tau^n} \left( O(t) \cdot c_t(\vec{x}, \tau) \cdot O^T(t) \right) \right|_{\tau=t} = O(t) \cdot \left. \frac{\partial^n}{\partial \tau^n} c_t(\vec{x}, \tau) \right|_{\tau=t} \cdot O^T(t),
\]

which, in view of (6.117), gives (6.118).

We again assume that all the functions in the argument of functional \( \mathcal{F}_t \) in (6.114) can be approximated by the truncated Taylor series expansion. Then the functional for a class of materials with bounded memory may be represented by functions involving various time derivative of the argument functions:

\[
t = \dot{t}(a_1, \ldots, a_{N_C}, \dot{\theta}, \ldots, \dot{\theta}^{(N_C)}, \ddot{\theta}, \ldots, \ddot{\theta}^{(N_C)}; \vec{g}, \ldots, \vec{g}^{(N_C)}),
\]

where the dependence on \( a_0 = I \) has been omitted because of its redundancy. Note that \( \dot{t} \) is again no longer functional, but a tensor-valued function. This is an asymptotic approximation of the most general constitutive equation (6.114) for a fluid. Coleman and Noll (1960) showed that this approximation is valid for sufficiently slow deformation processes.

Similar expressions hold for the heat flux vector \( \vec{q} \) and the internal energy \( \varepsilon \). The number of time derivatives for each of the gradients depends on the strength of the memory of these gradients. According to the axiom of equipresence, however, all constitutive functionals should be expressed in terms of the same list of independent constitutive variables.

### 6.16 Representation theorems for isotropic functions

We are now looking for the representation of a scalar-, vector- and tensor-valued function which are isotropic, that is, form-invariant for all orthogonal transformations.

The principle of fading memory, is then a requirement on smoothness of constitutive functionals. The principle of fading memory, mathematically formulated by Coleman and Noll (1960), starts with the assumption that the so-called Frechet derivatives of constitutive functionals up to an order \( n \) exist and are continuous in the neighborhood of histories at time \( t \) in the Hilbert space normed by an influence function of order greater than \( n + 1/2 \). Then, the constitutive functionals can be approximated by linear functionals for which explicit mathematical representations are known. The most important result of the fading memory theory is the possibility to approximate asymptotically a sufficiently slow strain and temperature histories by the Taylor series expansion.
6.16.1 Representation theorem for a scalar isotropic function

A scalar-valued function \( a \) depending on symmetric tensor \( S \) and vector \( \vec{v} \) is called isotropic if it satisfies the identity
\[
a(S, \vec{v}) = a(Q \cdot S \cdot Q^T, Q \cdot \vec{v})
\]  
(6.122)
for all orthogonal tensors \( Q \). This means that \( a \) does not depend on components \( S_{kl} \) and \( v_k \) arbitrarily, but it can only be considered as a function of scalar invariants that are independent of any orthogonal transformation.

Hence, we have to determine the set of all independent invariants of variables \( S \) and \( \vec{v} \). There are three independent invariants of tensor \( S \), namely,
\[
I_S = \text{tr} \, S \ , \quad II_S = \frac{1}{2} \left[ (\text{tr} \, S)^2 - \text{tr} \left( S^2 \right) \right] \ , \quad III_S = \det S \ ,
\]  
(6.123)
and one invariant of vector \( \vec{v} \), the scalar product \( \vec{v} \cdot \vec{v} \). Combining \( S \) and \( \vec{v} \), we can create a number of other invariants,
\[
\vec{v} \cdot S \cdot \vec{v} \ , \quad \vec{v} \cdot S^2 \cdot \vec{v} \ , \quad \vec{v} \cdot S^3 \cdot \vec{v} \ , \ldots , \, \vec{v} \cdot S^n \cdot \vec{v} \ , \ldots ;
\]  
(6.124)
of course, not all of them are mutually independent. According to the Cayley-Hamilton theorem saying that any matrix satisfies its characteristic polynomial, we have
\[
S^3 - I_S S^2 + II_S S - III_S I = 0 \ ,
\]  
(6.125)
where \( I \) and \( 0 \) are the unit and zero tensors of the same order as \( S \). Multiplying this equation from left and right by vector \( \vec{v} \), we obtain
\[
\vec{v} \cdot S^3 \cdot \vec{v} - I_S (\vec{v} \cdot S^2 \cdot \vec{v}) + II_S (\vec{v} \cdot S \cdot \vec{v}) - III_S (\vec{v} \cdot \vec{v}) = 0 \ .
\]  
(6.126)
We see that \( \vec{v} \cdot S^n \cdot \vec{v} \) for \( n \geq 3 \) can be expressed in terms of \( \vec{v} \cdot S^2 \cdot \vec{v}, \vec{v} \cdot S \cdot \vec{v} \) and \( \vec{v} \cdot \vec{v} \) but these cannot be used to express three invariants of \( S \). Hence, the set of all independent invariants of variables \( S \) and \( \vec{v} \) is
\[
I_S, II_S, III_S, \vec{v} \cdot \vec{v}, \vec{v} \cdot S \cdot \vec{v}, \quad \vec{v} \cdot S^2 \cdot \vec{v} \ .
\]  
(6.127)
We can conclude that scalar isotropic function \( a \) can be represented in the form
\[
a(S, \vec{v}) = a(I_S, II_S, III_S, \vec{v} \cdot \vec{v}, \vec{v} \cdot S \cdot \vec{v}, \vec{v} \cdot S^2 \cdot \vec{v}) \ .
\]  
(6.128)

6.16.2 Representation theorem for a vector isotropic function

A vector-valued function \( \vec{b} \) depending on symmetric tensor \( S \) and vector \( \vec{v} \) is called isotropic if it satisfies the identity
\[
Q \cdot \vec{b}(S, \vec{v}) = \vec{b}(Q \cdot S \cdot Q^T, Q \cdot \vec{v})
\]  
(6.129)
for all orthogonal tensors \( Q \). For a vector \( \vec{c} \), let us define function \( F \),
\[
F(S, \vec{v}, \vec{c}) := \vec{c} \cdot \vec{b}(S, \vec{v}) \ .
\]  
(6.130)
This function is a scalar isotropic function of variables \( S, \vec{v} \) and \( \vec{c} \) which follows from
\[
F(Q \cdot S \cdot Q^T, Q \cdot \vec{v}, Q \cdot \vec{c}) = Q \cdot \vec{c} \cdot \vec{b}(Q \cdot S \cdot Q^T, Q \cdot \vec{v}) = Q \cdot \vec{c} \cdot Q \cdot \vec{b}(S, \vec{v}) = \vec{c} \cdot Q^T \cdot Q \cdot \vec{b}(S, \vec{v}) = F(S, \vec{v}, \vec{c}) \ .
\]
In analogy to the preceding section, the set of all independent invariants of $S$, $\vec{v}$ and $\vec{c}$ is

$$I_S,\ II_S,\ III_S,\ \vec{v} \cdot \vec{v},\ \vec{v} \cdot S \cdot \vec{v},\ \vec{v} \cdot S^2 \cdot \vec{v},\ \vec{c} \cdot \vec{c},\ \vec{c} \cdot S \cdot \vec{c},\ \vec{c} \cdot S^2 \cdot \vec{c}.\quad (6.131)$$

If function $F$ was an arbitrary function of $S$, $\vec{v}$ and $\vec{c}$, it could be represented in terms of these 12 invariants in an arbitrary manner. Function $F$ is, however, linearly dependent on $\vec{c}$, so that it can be represented in terms of the invariants that are linear functions of $\vec{c}$ only:

$$F = a_0 \vec{c} \cdot \vec{v} + a_1 \vec{c} \cdot S \cdot \vec{v} + a_2 \vec{c} \cdot S^2 \cdot \vec{v},\quad (6.132)$$

where the coefficients $a_i$ depend on the six invariant that are independent of $\vec{c}$. Since $\vec{c}$ was an arbitrary vector in $(6.130)$–$(6.132)$, we can conclude that vector isotropic function $\vec{b}$ can be represented in the form

$$\vec{b} = a_0 \vec{v} + a_1 S \cdot \vec{v} + a_2 S^2 \cdot \vec{v},\quad (6.133)$$

where

$$a_i = a_i (I_S, II_S, III_S, \vec{v} \cdot \vec{v}, \vec{v} \cdot S \cdot \vec{v}, \vec{v} \cdot S^2 \cdot \vec{v} ).\quad (6.134)$$

6.16.3 Representation theorem for a symmetric tensor-valued isotropic function

(a) Symmetric tensor-valued isotropic function of symmetric tensor and vector. First, let us consider symmetric tensor-valued function $T$ depending on symmetric tensor $S$ and vector $\vec{v}$. Tensor $T$ is called isotropic if it satisfies the identity

$$Q \cdot T(S, \vec{v}) \cdot Q^T = T(Q \cdot S \cdot Q^T, Q \cdot \vec{v})\quad (6.135)$$

for all orthogonal tensors $Q$. For a tensor $A$, let us define function $F$,

$$F(S, \vec{v}, A) := \text{tr}(A \cdot T(S, \vec{v})) .\quad (6.136)$$

This function is a scalar isotropic function of variables $S$, $\vec{v}$ and $A$ which follows from

$$F(Q \cdot S \cdot Q^T, Q \cdot \vec{v}, Q \cdot A \cdot Q^T) = \text{tr}(Q \cdot A \cdot Q^T \cdot T(Q \cdot S \cdot Q^T, Q \cdot \vec{v}))$$

$$= \text{tr}(Q \cdot A \cdot Q^T \cdot Q \cdot T(S, \vec{v}) \cdot Q^T) = \text{tr}(Q \cdot A \cdot T(S, \vec{v}) \cdot Q^T) = \text{tr}(A \cdot T(S, \vec{v}) \cdot Q^T \cdot Q) = F(S, \vec{v}, A) .$$

Under the same argument as in the previous case, function $F$ must be represented as a linear combination of all independent invariants that are linear functions of $A$ only, that is, in terms of

$$\text{tr}A,\ \text{tr}(A \cdot S),\ \text{tr}(A \cdot S^2),\ \vec{v} \cdot A \cdot \vec{v},\ \vec{v} \cdot A \cdot S \cdot \vec{v},\ \vec{v} \cdot A \cdot S^2 \cdot \vec{v} .\quad (6.137)$$

Note that it is rather difficult to prove that there are no more independent invariants linearly dependent on $A$. Hence, we can write

$$F = a_0 \text{tr}A + a_1 \text{tr}(A \cdot S) + a_2 \text{tr}(A \cdot S^2) + a_3 \vec{v} \cdot A \cdot \vec{v} + a_4 \vec{v} \cdot A \cdot S \cdot \vec{v} + a_5 \vec{v} \cdot A \cdot S^2 \cdot \vec{v} ,\quad (6.138)$$

where the coefficients $a_i$ depend on the invariants that are independent of $A$, that is, on the six invariants of $S$ and $\vec{v}$. Because of the identity

$$\vec{v} \cdot A \cdot \vec{v} = \text{tr}(A \cdot \vec{v} \otimes \vec{v}) ,\quad (6.139)$$
where \( \vec{v} \otimes \vec{v} \) is the dyadic product of vector \( \vec{v} \) by itself, we also have

\[
F = a_0 \text{tr}A + a_1 \text{tr}(A \cdot S) + a_2 \text{tr}(A \cdot S^2) + a_3 \text{tr}(A \cdot \vec{v} \otimes \vec{v}) + a_4 \text{tr}(A \cdot S \cdot \vec{v} \otimes \vec{v}) + a_5 \text{tr}(A \cdot S^2 \cdot \vec{v} \otimes \vec{v}) .
\]  

(6.140)

The comparison of (6.140) with (6.136) results in the representation of a symmetric tensor-valued isotropic function \( T \) in the form

\[
T(S, \vec{v}) = a_0 I + a_1 S + a_2 S^2 + a_3 \vec{v} \otimes \vec{v} + a_4 \text{sym}(S \cdot \vec{v} \otimes \vec{v}) + a_5 \text{sym}(S^2 \cdot \vec{v} \otimes \vec{v}) ,
\]

where

\[
a_i = a_i (I_S, 1I_S, 111_S, \vec{v} \cdot \vec{v}, \vec{v} \cdot S \cdot \vec{v}, \vec{v} \cdot S^2 \cdot \vec{v}) \]  

(6.142)

and the symbol ‘sym’ stands for the symmetric part of tensor.

(b) Symmetric tensor-valued isotropic function of two symmetric tensors. Second, let us consider symmetric tensor-valued function \( T \) depending on two symmetric tensors \( A \) and \( B \). Tensor \( T \) is called isotropic if it satisfies the identity

\[
Q \cdot T(A, B) \cdot Q^T = T(Q \cdot A \cdot Q^T, Q \cdot B \cdot Q^T)
\]

(6.143)

for all orthogonal tensors \( Q \). For a tensor \( C \), let us define function \( F \),

\[
F(A, B, C) := \text{tr}(C \cdot T(A, B)) .
\]

(6.144)

This function must be a scalar isotropic function of variables \( A, B \) and \( C \). The set of all independent invariants of these three symmetric tensors that are linear functions of \( C \) is

\[
\text{tr}C, \text{tr}(C \cdot A), \text{tr}(C \cdot A^2), \text{tr}(C \cdot B), \text{tr}(C \cdot B^2), \text{tr}(C \cdot A \cdot B), \text{tr}(C \cdot A^2 \cdot B), \text{tr}(C \cdot A^2 \cdot B^2) .
\]

(6.145)

Hence, function \( F \) can be represented in the form

\[
F = a_0 \text{tr}C + a_1 \text{tr}(C \cdot A) + a_2 \text{tr}(C \cdot A^2) + a_3 \text{tr}(C \cdot B) + a_4 \text{tr}(C \cdot B^2) + a_5 \text{tr}(C \cdot A \cdot B) + a_6 \text{tr}(C \cdot A \cdot B^2) + a_7 \text{tr}(C \cdot A^2 \cdot B) + a_8 \text{tr}(C \cdot A^2 \cdot B^2) ,
\]

(6.146)

where the coefficients \( a_i \) depend on the invariants that are independent of \( C \), that is, on the ten invariants of \( A \) and \( B \):

\[
I_A, II_A, III_A, I_B, II_B, III_B, \text{tr}(A \cdot B), \text{tr}(A \cdot B^2), \text{tr}(A^2 \cdot B), \text{tr}(A^2 \cdot B^2) .
\]

(6.147)

The comparison of (6.146) with (6.144) yields the representation of a symmetric tensor-valued isotropic function \( T \) in the form

\[
T(A, B) = a_0 I + a_1 A + a_2 A^2 + a_3 B + a_4 B^2 + a_5 \text{sym}(A \cdot B) + a_6 \text{sym}(A \cdot B^2) + a_7 \text{sym}(A^2 \cdot B) + a_8 \text{sym}(A^2 \cdot B^2) ,
\]

(6.148)

where

\[
a_i = a_i (I_A, II_A, III_A, I_B, II_B, III_B, \text{tr}(A \cdot B), \text{tr}(A \cdot B^2), \text{tr}(A^2 \cdot B), \text{tr}(A^2 \cdot B^2)) .
\]

(6.149)

6.17 Examples of isotropic materials with bounded memory
6.17.1 Elastic solid

A solid is called *elastic* if the stress tensor $t$ depends only on the deformation gradient $F$ at the present time, not on the temperature $\theta$ and not on the entire past thermomechanical history:

$$t = \hat{t}(F) .$$

Hence, the stress in an elastic material at each particle is uniquely determined by the present deformation from a fixed reference configuration. The material objectivity requires that the dependence on $F$ is not arbitrary, but it has the form (6.29):

$$t = R \cdot \hat{t}(C) \cdot R^T .$$

For isotropic materials, $\hat{t}$ must be an isotropic function of $C$. Making use of the representation theorem for a tensor isotropic function, we have

$$t = R \cdot (a_0 I + a_1 C + a_2 C^2) \cdot R^T ,$$

where $a_i$ are scalar functions of three principal invariants of $C$,

$$a_i = a_i(I_C, II_C, III_C) .$$

With the help of the Finger deformation tensor $b$ and its second power,

$$b = F \cdot F^T = V^2 = V^T \cdot V = R \cdot F^T \cdot F \cdot R^T = R \cdot C \cdot R^T ,$$

$$b^2 = R \cdot C \cdot R^T \cdot R \cdot C \cdot R^T = R \cdot C^2 \cdot R^T ,$$

we obtain a general form of the constitutive equation of an isotropic elastic material:

$$t = a_0 I + a_1 b + a_2 b^2 , \quad a_i = a_i(I_b, II_b, III_b) .$$

This does not imply that the constitutive equation can only be a quadratic in the components of $b$, since scalar functions $a_i$ may be non-linear in terms of the components of $b$.

The constitutive equation (6.34) for the second Piola-Kirchhoff stress tensor for an elastic material (even without isotropy) is reduced to the simple form:

$$T^{(2)} = \hat{T}^{(2)}(C) .$$

The function $\hat{T}^{(2)}$ for isotropic materials can be represented in the form:

$$T^{(2)} = a_0 I + a_1 C + a_2 C^2 , \quad a_i = a_i(I_C, II_C, III_C) ,$$

which is another general form, equivalent to (6.154), of the constitutive equation for isotropic elastic materials.

We now have a look at the case of small deformation when the displacement gradient $H$ is sufficiently small such that the geometric linearization can be applied and the difference between the reference configuration and the present configuration is not necessary to consider. Within the limit of geometric linearization, the Finger deformation tensor is equal to the Green deformation tensor. Consequently, the linearized constitutive equation for the second Piola-Kirchhoff tensor
coincides with that for the Cauchy stress tensor. Let us, for instance, linearize the constitutive equation (6.156). Because of (1.106), we have

\[
C = U^2 = I + 2\dot{E}, \quad C^2 = I + 4\dot{E}, \quad I_C = 3 + 2\text{tr}\dot{E}, \quad II_C = 3 + 4\text{tr}\dot{E}, \quad III_C = 1 + 2\text{tr}\dot{E},
\]

(6.157)

where \( \dot{E} \) is the infinitesimal strain tensor. Then the constitutive equations (6.156) and (6.154) may be linearized as

\[
T(2) \cong t = \lambda(\text{tr}\dot{E})I + 2\mu\dot{E}.
\]

(6.158)

This is the constitutive equation of an isotropic, linear elastic material, known also as Hook’s law. The both parameters \( \lambda \) and \( \mu \) are called the Lamé elastic constants.

6.17.2 Thermoelastic solid

The effect of heat conduction can be incorporated in the behavior of elastic materials by including temperature and temperature gradients among the constitutive variables. From (6.117) we deduce that the classical (non-classical) thermoelastic solid is defined by

\[
N_C = 0, \quad N_\theta = 0(1) \quad \text{and} \quad N_g = 0.
\]

Hence, the constitutive equation for a thermoelastic solid will have the form:

\[
T^{(2)} = \hat{T}^{(2)}(C, \theta, \dot{\theta}, \ddot{G}_\theta), \quad \ddot{Q} = \hat{\ddot{Q}}(C, \theta, \dot{\theta}, \ddot{G}_\theta), \quad \varepsilon = \hat{\varepsilon}(C, \theta, \dot{\theta}, \ddot{G}_\theta).
\]

(6.159)

For isotropic materials, functions \( \hat{T}^{(2)}, \hat{\ddot{Q}} \) and \( \hat{\varepsilon} \) must be isotropic functions of their arguments:

\[
\begin{align*}
T^{(2)} &= a_0 I + a_1 C + a_2 C^2 + a_3 \ddot{G}_\theta \otimes \ddot{G}_\theta + a_4 \text{sym}(C \cdot \ddot{G}_\theta \otimes \ddot{G}_\theta) + a_5 \text{sym}(C^2 \cdot \ddot{G}_\theta \otimes \ddot{G}_\theta), \\
\ddot{Q} &= -\kappa \dddot{G}_\theta + c_1 C \cdot \dddot{G}_\theta + c_2 C^2 \cdot \dddot{G}_\theta, \\
\varepsilon &= \hat{\varepsilon}(\theta, \dot{\theta}, I_C, II_C, III_C, \dddot{G}_\theta \cdot \dddot{G}_\theta, \dddot{G}_\theta \cdot C \cdot \dddot{G}_\theta, \dddot{G}_\theta \cdot C^2 \cdot \dddot{G}_\theta),
\end{align*}
\]

(6.160)

where the scalar function \( a_i, \kappa \) and \( c_i \) depend on the same set of arguments as the function \( \hat{\varepsilon} \) of the internal energy. We can see that the heat flux vector \( \dddot{Q} \) depends linearly on the temperature gradient \( \dddot{G}_\theta \) but contains two non-linear terms as the Green deformation tensor arises together with \( \dddot{G}_\theta \). The coefficient \( \kappa \) is called the heat conductivity, the remaining coefficients \( c_1 \) and \( c_2 \) do not have separate names.

6.17.3 Kelvin-Voigt viscoelastic solid

In general, under the viscoelastic solid we understand the material for which the stress is dependent on the strain, strain rate and stress rate tensors and not on the temperature. From (6.117) we deduce that the simplest viscoelastic solid is defined by \( N_C = 1 \):

\[
T^{(2)} = \hat{T}^{(2)}(C, \dot{C}).
\]

(6.161)

This simple strain-rate dependent solid is called the Kelvin-Voigt viscoelastic solid. Since the Piola-Kirchhoff stress tensor depends on two symmetric tensors, for isotropic materials it may be represented according to (6.148):

\[
T^{(2)} = a_0 I + a_1 C + a_2 C^2 + a_3 \dot{C} + a_4 \dot{C}^2 + a_5 \text{sym}(C \cdot \dot{C}) + a_6 \text{sym}(C \cdot \dot{C}^2) + a_7 \text{sym}(C^2 \cdot \dot{C}) + a_8 \text{sym}(C^2 \cdot \dot{C}^2),
\]

(6.162)
where
\[
a_i = a_i(\text{tr} C, \text{tr} C^2, \text{tr} C^3, \text{tr} \dot{C}, \text{tr} \dot{C}^2, \text{tr}(C \cdot \dot{C}), \text{tr}(C^2 \cdot \dot{C}), \text{tr}(C^2 \cdot \dot{C}^2)) .
\] (6.163)

### 6.17.4 Maxwell viscoelastic solid

A special case of constitutive equation (6.6) expressed in terms of the second Piola-Kirchhoff stress tensor is
\[
(T^{(2)})^* = \hat{T}^{(2)}(T^{(2)}, \dot{C}^*) .
\] (6.164)

This simple stress-rate dependent solid is called the **Maxwell viscoelastic solid**. Since \(\hat{T}^{(2)}\) is an objective scalar, see (6.18)_2, and the material time derivative of an objective scalar is again an objective scalar, the constitutive equation (6.164) satisfies the principle of material objectivity. If \(\hat{T}^{(2)}\) is an isotropic function, we can proceed similarly as for the Kelvin-Voigt viscoelastic solid.

### 6.17.5 Elastic fluid

An elastic material may be a solid or a fluid. For an **elastic fluid**, the functions in the argument of the constitutive function \(\hat{t}\) in (6.121) are reduced to the dependence on the current density \(\varrho\):
\[
t = \hat{t}(\varrho) .
\] (6.165)

According to the assumption of material objectivity, we have
\[
t^* = \hat{t}(\varrho^*)
\] (6.166)
under a change of spatial frame. Because \(\varrho^* = \varrho\) and \(t^* = O \cdot t \cdot O^T\), this implies that
\[
O \cdot t(\varrho) \cdot O^T = \hat{t}(\varrho)
\] (6.167)
for all orthogonal tensors \(O\). The only tensor satisfying this identity is a spherical tensor:
\[
t = -p(\varrho)I .
\] (6.168)
Therefore the stress in an elastic fluid is always a hydrostatic pressure depending on the density alone.

### 6.17.6 Thermoelastic fluid

The effect of heat conduction can be incorporated in the behavior of elastic fluid by including temperature and temperature gradients among the constitutive variables. From (6.121) we deduce that the **classical (non-classical) thermoelastic (heat-conducting) fluid** is defined by \(N_C = 0\), \(N_\theta = 0(1)\) and \(N_g = 0\). Hence, the constitutive equation for a thermoelastic fluid will have the form:
\[
t = \hat{t}(\theta, \dot{\theta}, \vec{g}_\theta, \varrho), \quad \vec{q} = \hat{q}(\theta, \dot{\theta}, \vec{g}_\theta, \varrho), \quad \varepsilon = \hat{\varepsilon}(\theta, \dot{\theta}, \vec{g}_\theta, \varrho) .
\] (6.169)

Making use of the representation theorems for isotropic functions, we obtain
\[
t = \sigma I + \tau \vec{g}_\theta \otimes \vec{g}_\theta ,
\]
\[
\vec{q} = -\kappa \vec{g}_\theta ,
\]
\[
\varepsilon = \varepsilon(\varrho, \theta, \dot{\theta}, \vec{g}_\theta \cdot \vec{g}_\theta) ,
\] (6.170)
where
\[ \sigma, \tau, \kappa = \sigma, \tau, \kappa(\varrho, \theta, \hat{\theta}, \vec{g}_\theta \cdot \vec{g}_\theta) . \] (6.171)

### 6.17.7 Viscous fluid

Taking \( N_C = 1 \) in (6.121), we obtain a *viscous fluid*, an analog to a simple viscoelastic solid:

\[ t = \hat{t}(a_1, \varrho) = \hat{t}(\varrho, I_d, I_d, II_d) , \] (6.172)

since \( a_1 = 2d \). This constitutive equation can further be simplified by applying the representation theorem for a symmetric isotropic tensor:

\[ t = a_0 I + 2\mu_v d + a_2 d^2 , \quad a_i, \mu_v = a_i, \mu_v(\varrho, I_d, II_d, III_d) , \] (6.173)

where \( \mu_v \) is the so-called *(shear) viscosity*.

Now, we wish to show that the constitutive equation (6.173) is materially objective. Multiplying (6.173) from the left by \( O \) and from the right by \( O^T \), we obtain

\[ O \cdot t \cdot O^T = a_0 O \cdot I \cdot O^T + 2\mu_v O \cdot d \cdot O^T + a_2 O \cdot d \cdot O^T \cdot O \cdot d \cdot O^T , \quad a_i, \mu_v = a_i, \mu_v(\varrho, I_d, II_d, III_d) . \] (6.174)

Since \( t \) and \( d \) are objective tensors and \( \varrho \) as well as the invariants of \( d \) are objective scalars, it holds

\[ t^* = a_0 I + 2\mu_v d^* + a_2 (d^*)^2 , \quad a_i, \mu_v = a_i, \mu_v(\varrho^*, I_d^*, II_d^*, III_d^*) , \] (6.175)

which is of the same form as that in the unstarred frame. This completes the proof.

### 6.17.8 Incompressible viscous fluid

For an incompressible fluid we have the kinematic constraint that the density is constant, equal to a known value: \( \varrho = \varrho_0 \). The mass-conservation principle then yields: \( I_d = \text{tr} d = \text{div} \vec{v} = 0 \). Combining (6.54) and (6.121), the constitutive equation of an incompressible viscous fluid is

\[ t = -p I + t^D . \] (6.176)

The pressure \( p \) is an additional unknown independent field variable and it would be determined through solutions of field equations under a given set of boundary conditions. On contrary, the deviatoric part \( t^D \) of the stress tensor \( t \) is given by a constitutive equation analogous to (6.173):

\[ t^D = 2\mu_v d + a_2 d^2 , \quad a_i, \mu_v = a_i, \mu_v(II_d, III_d) . \] (6.177)

The dependencies on \( \varrho \) and \( I_d \) are dropped since \( \varrho \) is a known constant and \( I_d = 0 \). Notice that constitutive relation (6.176) has been postulated in such a way that any spherical part of stress tensor \( t \) may, without loss of generality, be absorbed in the pressure.\(^6\)

\(^6\)The stress tensor \( t \) may, in general, be decomposed into *isotropic spherical tensor* \( \sigma I \) and the *deviatoric stress tensor* \( t^D \):

\[ t = \sigma I + t^D . \]

Under the choice of \( \sigma = \text{tr} t/3 \), the trace of the deviatoric stress vanishes, \( \text{tr} t^D = 0 \). The scalar \( \sigma \) is thus the mean of the normal-stress components and is called *mechanical pressure*. A characteristic feature of a fluid at rest is that it cannot support shear stresses. Consequently, the deviatoric stress identically vanishes. Choosing \( p := -\sigma \), we obtain \( t = -p I \). Therefore, the stress in a fluid at rest is the so-called *hydrostatic pressure*.  

31
6.17.9 Viscous heat-conducting fluid

A classical (non-classical) viscous heat-conducting fluid is defined by $N_C = 1$, $N_\theta = 0(1)$ and $N_\vartheta = 0$, so that (6.121) reduces to

$$t = l(d, \theta, \dot{\theta}, g_\theta, \vartheta), \quad \vec{q} = \vec{q}(d, \theta, \dot{\theta}, g_\theta, \vartheta), \quad \varepsilon = \hat{\varepsilon}(d, \theta, \dot{\theta}, g_\theta, \vartheta).$$  (6.178)

Making use of the representation theorems for isotropic functions, we obtain

$$t = a_0 I + 2 \mu_v d + a_2 d^2 + a_3 g_\theta \otimes g_\theta + a_4 \text{sym}(d \cdot g_\theta \otimes g_\theta) + a_5 \text{sym}(d^2 \cdot g_\theta \otimes g_\theta),$$

$$\vec{q} = -\kappa \vec{g}_\theta + c_1 d \cdot \vec{g}_\theta + c_2 d^2 \cdot \vec{g}_\theta,$$

$$\varepsilon = \hat{\varepsilon}(\varrho, \theta, \dot{\theta}, I_d, II_d, III_d, g_\theta \cdot \vec{g}_\theta, \vec{g}_\theta \cdot d, \vec{g}_\theta \cdot d^2, \vec{g}_\theta),$$  (6.179)

where the scalars $a_i$, $\mu_v$, $\kappa$ and $c_i$ depend on the same set of arguments as the function $\hat{\varepsilon}$ of the internal energy. The proof of the material objectivity of these constitutive equations is similar to that for the viscous fluid.

Fluids characterized by the above non-linearity are called Stokesian or non-Newtonian fluids. On the other hand, linear behavior is generally referred to as Newtonian behavior. If, moreover, the constitutive equations are also linear with respect to the temperature gradient, the emerging field equations are called the Navier-Stokes equations.