Dynamic stress field of a kinematic earthquake source model with $k$-squared slip distribution

Jan Burjánek and Jiří Zahradník
Charles University in Prague, Faculty of Mathematics and Physics, Department of Geophysics, V Holesovickach 2, 18000 Praha 8, Czech Republic. E-mail: burjanek@karel.troja.mff.cuni.cz

Accepted 2007 July 7. Received 2007 April 26; in original form 2006 August 19

SUMMARY
Source models such as the $k$-squared stochastic source model with $k$-dependent rise time are able to reproduce source complexity commonly observed in earthquake slip inversions. An analysis of the dynamic stress field associated with the slip history prescribed in these kinematic models can indicate possible inconsistencies with physics of faulting. The static stress drop, the strength excess, the breakdown stress drop and critical slip weakening distance $D_c$ distributions are determined in this study for the kinematic $k$-squared source model with $k$-dependent rise time. Several studied $k$-squared models are found to be consistent with the slip weakening friction law along a substantial part of the fault. A new quantity, the stress delay, is introduced to map areas where the yielding criterion of the slip weakening friction is violated. Hisada’s slip velocity function is found to be more consistent with the source dynamics than Boxcar, Brune’s and Dirac’s slip velocity functions. Constant rupture velocities close to the Rayleigh velocity are inconsistent with the $k$-squared model, because they break the yielding criterion of the slip weakening friction law. The bimodal character of $D_c/D_{tot}$ frequency–magnitude distribution was found. $D_c$ approaches the final slip $D_{tot}$ near the edge of both the fault and asperity. We emphasize that both filtering and smoothing routinely applied in slip inversions may have a strong effect on the space–time pattern of the inferred stress field, leading potentially to an oversimplified view of earthquake source dynamics.

Key words: fault model, fault slip, rupture propagation, source time functions, stress distribution.

1 INTRODUCTION
Realistic and accurate estimation of strong ground motion in a broad frequency range for future large earthquakes is one of the major topics of present seismology. A realistic earthquake source model should form an integral part of every study concerning strong ground motions in the proximity of an active fault. Recently, advanced theoretical kinematic source models have been developed including stochastic distributions of certain source parameters. The most established ones are the composites source model with the fractal subevent size distribution of Zeng et al. (1994) and the $k$-squared model of Herrero & Bernard (1994), further developed in the papers of Bernard et al. (1996), Hisada (2000, 2001) and Gallovic & Brokesova (2004). Such models simulate the space–time evolution of slip on a fault, and their radiated wavefield follows the widely accepted $\omega$-squared model. They were successfully applied in a number of strong motions studies (e.g. Zollo et al. 1997; Berge-Thierry et al. 2001; Emolo & Zollo 2001).

Concurrently, great effort has been put into the dynamic modelling of earthquake sources; an overview can be found in Madariaga & Olsen (2002). A number of dynamic models of recent earthquakes were also developed (e.g. Quin 1990; Olsen et al. 1997), since a number of good quality kinematic earthquake source inversions have been obtained and computer power increased. Several attempts have been made to apply these new findings in forward modelling of rupture propagation in stochastic stress fields, to provide physical scenarios for possible future earthquakes (Oglesby & Day 2002; Guattieri et al. 2003). Such studies are very valuable since they introduce more physics into the problem, but there are still many open questions concerning possible strength and stress distributions which are generally unknown. Problems are also connected with the computation itself, which is still very time consuming and thus not very convenient for seismic hazard assessment. Another approach is represented by the so-called pseudo-dynamic model of Guattieri et al. (2004), who constrain the parameters of the theoretical kinematic model by relations obtained from forward dynamic simulations.

In the present study, we follow an alternative way. We investigate the stress field generated by theoretical $k$-squared kinematic models. Previously, Madariaga (1978) studied the dynamic stress field of Haskell’s fault model and found strong contradictions with earthquake source physics (e.g. an infinite average static stress drop).
Analogous studies on earthquake source dynamics using kinematic source inversions were presented by Bouchon (1997), Ide & Takeo (1997), Day et al. (1998), Dalguer et al. (2002) and Piatanesi et al. (2004). These studies led to estimates of fault friction parameters. In this study, we calculate the dynamic stress field caused by a slip history prescribed by the \( k \)-squared kinematic model. The combination of these stress changes and prescribed slip time-series implies a constitutive relation at every point along the fault, and we ask whether these constitutive relations are ‘reasonable’. Specifically, we confront these constitutive relations with the sliding weakening friction law. We analyse the strength excess, breakdown stress drop, and critical slip weakening distance \( D_c \) distributions. A new parameter, the stress delay \( T_x \), is introduced to map the fault points where the peak stress precedes the rupture onset—points where implied constitutive relations violate the slip weakening friction law. We choose the following criteria for the ‘reasonable’ constitutive relation: (1) minimal areas of stress delay \( T_x \) (less than 5 per cent of the fault area) and (2) minimal areas of negative strength excess (less than 5 per cent of the fault area). Additionally, we require a small ratio of \( D_c \) to total slip. If a kinematic source model fails to fulfil these criteria, we suggest its rejection. The choice of 5 per cent is arbitrary.

Another point of this paper is the analysis of effects of space–time filtering on resulting dynamic stress field, which is important for the correct interpretation of band-limited slip inversions of real earthquakes.

### 2 Method of Calculation

To determine the dynamic stress history on a fault from the space–time slip distribution prescribed by the kinematic model we have used the boundary element method proposed by Bouchon (1997). The shear stress change is expressed as the time convolution of the slip time function and stress Green’s function which is calculated using the discrete wavenumber method, particularly by computing the Weyl-like integral by a 2-D discrete Fourier series. The time convolution is performed in the spectral domain. Since, we are working with synthetic data, we assume a rectangular fault situated in an infinite isotropic homogeneous space, approximating a buried fault. We have followed the solution presented in Bouchon (1997), except that we have employed slip velocity functions (instead of slip functions) and performed the time integration of the stress time histories at the end of the procedure. Both slip and stress time histories have non-zero static parts. We cannot simply apply FFT, it would lead to alias effect in the time domain. Thus it is convenient to work with the slip velocity functions, getting stress rates. Hence, we can perform frequency filtering easily and apply FFT algorithm for the inverse Fourier transform to get the result in the time domain. The method of calculation can be extended to non-planar faults in a layered half-space, however, such calculations would be computationally very expensive and a finite difference or finite element method would be more appropriate.

### 3 k-Squared Model with Asperities

Let us shortly introduce the \( k \)-squared model with asperities, which is the subject of our study. The basis of all kinematic \( k \)-squared models is a stochastic final slip distribution, which is defined as follows: the amplitude spectrum of the static slip distribution is flat up to a certain characteristic corner wavenumber, then falling off as the inverse power of two, thus matching the condition of self-similarity (Andrews 1980; Herrero & Bernard 1994). The corner wavenumber represents the fault roughness. Attempts were made to estimate this wavenumber and spectral falloff from the static slip distributions obtained by kinematic source inversions (Somerville et al. 1999; Mai & Beroza 2002). However, such results may be biased by the smoothing procedures common to most slip inversions and hence one has to be very careful in taking them into account.

In the paper of Somerville et al. (1999), an attempt was made to investigate the heterogeneity of static slip distribution directly in the space domain, defining asperities as regions covering some minimum predefined areas, where the average slip exceeds the prescribed limit. The asperities should represent the behaviour of slip models at low wavenumbers. Since the asperities seem to be the dominant regions of the earthquake source in seismic wave generation (Miyake et al. 2003), synthetic slip models for strong ground motion prediction should mimic asperities. The total area of asperities and the area of the greatest asperity exhibit clear seismic moment dependence and thus can be estimated for future earthquakes of a given magnitude. The position of asperities within the fault is difficult to predict, although attempts were made to link the centre of the largest asperity with the position of hypocentre (Somerville et al. 1999; Mai et al. 2005). Alternatively, attempts have been made to verify the hypothesis of permanent asperities (Irikura, private communication, 2003), who proposes that the asperity always takes the same place on a particular fault.

There are several ways of generating a stochastic \( k \)-squared static slip distribution. Most common is the method of filtering noise (Andrews 1980; Herrero & Bernard 1994; Somerville et al. 1999; Mai & Beroza 2002; Gallovic & Brokoseva 2004). The random phase generator is usually based on a uniform probability distribution. Such synthetic slip models do not generally provide a direct control of the asperities. Lavalle & Archuleta (2003) applied Lévy probability density function to pronounce asperities. Another way is to assume the asperities in the space domain and add high wavenumber noise with given properties (Gallovic & Brokoseva 2004). Hence one has a direct control of asperities—one can prescribe the size, the seismic moment and the position of the asperities. It is especially useful in cases when the smooth slip distribution is known (i.e. the properties of asperities). Then the broad-band source model can be easily created by adding the high wavenumber stochastic component. Particularly, the 2-D Fourier spectrum of the slip distribution reads

\[
U(k_x, k_y) = \frac{\Delta \bar{u} L W}{\sqrt{1 + \left( \frac{k_x L}{K_c} \right)^2 + \left( \frac{k_y W}{K_c} \right)^2}} e^{i \Phi(k_x, k_y)}, \tag{1}
\]

where \( k_x, k_y \) are horizontal wavenumbers (in the fault plane), \( \Delta \bar{u} \) is the average slip, \( L \) is the length and \( W \) is the width of the fault, \( K_c \) is a dimensionless constant—the relative corner wavenumber, \( \Phi \) is the random phase function of \( k_x, k_y \). Note that the amplitude spectrum has the form of a low-pass Butterworth filter with the cut-off wavenumber controlled by \( K_c \). By changing \( K_c \) we can then demonstrate the effect of spatial filtering of the final slip distribution.

Another approach is represented by superposing slip patches with an average slip proportional to the size of the patch, such that the overall slip provides \( k \)-squared falloff at high wavenumbers (Frankel 1991; Zeng et al. 1994; Gallovic & Brokoseva 2007). The position of these patches is random, except for the largest ones, which can be employed to build up an asperity.

We have used both types of \( k \)-squared slip generators: the one described in Gallovic & Brokoseva (2004) and the one described in...
Gallow & Brookesova (2007). We assume scalar seismic moment \( M_0 = 7.8 \times 10^{17} \) Nm released along a rectangular fault, size 11 \( \times \) 8 km located in an infinite, homogeneous, isotropic, elastic space characterized by \( P \)-wave velocity \( v_p = 6 \) km s\(^{-1}\), \( S \)-wave velocity \( v_s = 3.46 \) km s\(^{-1}\) and density \( \rho = 2800 \) kg m\(^{-3}\). We also assume a rectangular asperity in one corner of the fault with an average asperity slip contrast of 2 following Somerville et al. (1999). The direction of the slip vector is constant (parallel to the fault length over the whole fault) to make the analysis easier. Four different slip distributions are shown in Fig. 1(a), which all include an asperity with the same properties—rectangular quadrant of the fault with its average slip two times larger than the average slip along the whole fault. One slip distribution was generated by the patch method (PM distribution, first from the left) with the largest subevent of the asperity size. The other three were generated by filtering white noise.

The three slip distributions differ only in their relative corner wavenumber \( K_c \) (\( K_c = 1, 0.75 \) and \( K_c = 0.5 \)). All the three were created by filtering white noise (\( K_c = \infty \)). One can see the effect of \( K_c \)—the higher \( K_c \), the rougher the slip distribution. We can interpret alternatively the two slip distributions (\( K_c = 0.75 \) and 0.5) as the low-pass filtered versions of the \( K_c = 1 \) slip distributions. Thereby we demonstrate the effect of spatial filtering, which is naturally present in kinematic slip inversions.

**4 STATIC STRESS FIELD**

First, we analyse the static stress change along the fault due to the slip distributions described in the previous section. A static stress change due to the self-similar rupture model with the \( k \)-squared slip distribution was already studied by Andrews (1980). We extend the study to the \( k \)-squared model with an asperity and also for different slip roughnesses. As the fault is planar, the normal stress change along the fault is zero. We thus calculated just the shear traction change expressed in slip-parallel and slip-perpendicular components. The slip-parallel component dominates over the slip-perpendicular component along most of the fault. However, in some sections of the fault both components are comparable. Thus the shear traction change there can become even perpendicular to the slip vector. Nevertheless, the magnitudes of these shear stress changes are lower compared to the rest of the fault (up to 10 per cent of the peak shear stress changes). Quantitatively, the shear traction deviates by less than \( \pm 45^\circ, \pm 15^\circ \) and \( \pm 10^\circ \) from the slip-parallel direction over 94, 75 and 61 per cent of the fault area, respectively. Hereafter, we work just with the slip-parallel component, assuming an absolute stress level much greater than the stress change (Guatteri & Spudich 1998). We distinguish the stress drop (or positive stress drop) and negative stress drop if the stress change points along or in the opposite direction of the slip vector, respectively. The results are depicted in Fig. 1(b). One can see the areas of both the positive and negative stress drop, which are separated by white contours. Remember that in the vicinity of these contours the shear traction change is not well represented by the slip-parallel component. As the PM slip distribution is smoother, the resulting static stress change appears to be simpler, and almost all the areas of stress drop are connected. On the other hand, the static stress change due to the \( K_c = 1 \) slip distribution is very complex and areas of stress drop create isolated islands. Some characteristics of the static stress change are summarized in Table 1. The average stress change is the same for all \( K_c \) slip distributions, but the spatial variability of the static stress change and the area of the negative stress drop increase with increasing \( K_c \).

**Figure 1.** Final slip distributions (a) and slip-parallel static shear stress changes (b). The first slip (from the left) was generated by the patch method (PM), other slip distributions were generated by filtering white noise for different values of corner wavenumber \( K_c \). The white contours in the stress pictures separate positive and negative static stress drop areas.
Table 1. Some basic parameters of the static stress field due to the \( k \)-squared slip distributions of Fig. 1. \( \Delta \sigma_{\text{min}} \) and \( \Delta \sigma_{\text{max}} \) are ranges of the stress changes along the fault, \( \langle \Delta \sigma \rangle \) is the average stress change over the fault, \( \langle \Delta \sigma^+ \rangle \) is the average positive stress drop and \( A^{\Delta \sigma^+}/A^{\text{fault}} \) is the ratio of the positive stress drop area to the whole fault area.

<table>
<thead>
<tr>
<th>Static stress changes</th>
<th>( K_x = 1 )</th>
<th>( K_x = 0.75 )</th>
<th>( K_x = 0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta \sigma_{\text{min}} ) (MPa)</td>
<td>-28</td>
<td>-75</td>
<td>-41</td>
</tr>
<tr>
<td>( \Delta \sigma_{\text{max}} ) (MPa)</td>
<td>11</td>
<td>55</td>
<td>33</td>
</tr>
<tr>
<td>( \langle \Delta \sigma \rangle ) (MPa)</td>
<td>-1.6</td>
<td>-1.6</td>
<td>-1.6</td>
</tr>
<tr>
<td>( \langle \Delta \sigma^+ \rangle ) (MPa)</td>
<td>4.7</td>
<td>16.2</td>
<td>8.8</td>
</tr>
<tr>
<td>( A^{\Delta \sigma^+}/A^{\text{fault}} ) (per cent)</td>
<td>60</td>
<td>45</td>
<td>49</td>
</tr>
</tbody>
</table>

5 DYNAMIC STRESS FIELD

If we want to study the dynamic stress field we have to prescribe the slip time history. We have employed Dirac’s delta function and slip velocity functions with the so-called \( k \)-dependent rise time as, together with the \( k \)-squared slip distribution (Herrero & Bernard 1994; Bernard et al. 1996), they generate the widely accepted \( k \)-squared model. Gallovic & Brokesova (2004) generalized the concept of the \( k \)-squared slip function (Herrero & Bernard 1994; Bernard et al. 1996), they generate the widely accepted \( k \)-squared model. Gałłowicz & Brokesova (2004) generalized the concept of the \( k \)-dependent rise time for the general slip velocity function (SVF), so that the spectrum of the slip function \( \Delta \hat{u}(x, \omega) \) can be expressed as:

\[
\Delta \hat{u}(x, \omega) = \frac{1}{\omega} \int \frac{\omega}{2\pi} U(k) \hat{X}(k) \, dk,
\]

where \( \omega \) is the angular frequency, \( x \) is the position along the fault, \( x_{\text{hyp}} \) is the position of the nucleation point, \( v_r \) is the rupture velocity, \( k \) is the horizontal wave vector (in the fault plane), \( U(k) \) is the 2-D Fourier spectrum of the static slip distribution, \( \hat{X}(k) \) is the spectrum of the slip function, \( \tau(k) \) is the \( k \)-dependent rise time, \( \tau_{\text{max}} \) is the maximum rise time, and \( a \) is the non-dimensional coefficient described in Bernard et al. (1996), where the authors suggested \( a = 0.5 \). The rupture velocity \( v_r \) has to be constant. We have performed the calculation for three SVFs with \( k \)-dependent rise time—Boxcar, Brune’s function and the Kostrov-like function proposed by Hisada (2000) (we have thus denoted it Hisada’s function) with maximum rise time \( \tau_{\text{max}} = 1 \) s. We have taken \( v_r = 2.6 \) km s\(^{-1}\) (\( = 0.75 v_s \)). The slip is thus propagating in a pulse of half width \( L_0 = 2.6 \) km in case of the SVF with \( k \)-dependent rise time. The nucleation point was chosen in the middle of the fault left border—just at the corner of the asperity (indicated later by an asterisk in Fig. 4).

An example of the stress histories at a point along the fault for four different SVFs is shown in Fig. 2. One can recognize the onset due to \( P \) waves, and the peaks associated with the \( S \) wave and rupture front arrivals, respectively, for all types of SVFs. The static part of the stress histories is independent of SVF. There is a clear singularity associated with the rupture front arrival in case of Dirac’s SVF. Brune’s SVF is similar to Hisada’s SVF, thus the stress time histories are also very similar. We also show the effect of frequency filtering, the stress time histories being plotted for three different cut-off frequencies. As we are working with step-like functions, we use a filter whose Fourier amplitude spectrum falls off very slowly to overcome the spurious overshoots in the signal, that is, a cosine window \( 0.5[\cos(0.5\omega/\tau_{\text{max}}) + 1] \). The filter is acausal. Note the strong effect of the frequency filtering on the peak stress associated with the rupture onset.

![Figure 2](https://example.com/figure2.png)

Figure 2. Stress time histories for different shapes of slip velocity functions (Dirac’s, Boxcar, Brune’s and Hisada’s) at a point of the fault. The slip functions were generated for the slip distribution denoted as PM in Fig. 1. To demonstrate the effect of filtering we applied the low-pass filter changing cut-off frequency \( f_{\text{max}} \) gradually (3, 6 and 12 Hz). The origin of the time axis is arbitrary and the plots are additionally shifted for clarity by the same amount according to the slip velocity function. The plot is clipped in the case of Dirac’s slip velocity function.
Stress field of the $k$-squared model

Figure 3. Four representative stress change time histories. Reading of the strength excess, dynamic stress drop, breakdown stress drop and definition of stress delay $T_x$ is pointed out for each time history. Cases (a) and (b) represent time histories with peak stress preceding rupture front onset. Case (b) represents the case with negative strength excess. Case (c) represents the time history with immediate stress increase just after the rupture onset—with zero breakdown stress drop. Case (d) represents the time history with negative static stress drop but with positive breakdown stress drop.

For further quantitative analysis of the dynamic stress field we determine several dynamic source characteristics—strength excess (SE), dynamic stress drop (DS), breakdown stress drop (BS) and stress delay $T_x$. The definition of these quantities for the representative set of stress time histories is shown in Fig. 3. The strength excess (SE) is the value of the stress level at the very onset of the rupture (e.g. Bouchon 1997). The dynamic stress drop (DS) is defined as the minimal stress level after rupture arrival (e.g. Dalguer et al. 2002). The breakdown stress drop (BS) is the sum of the strength excess and the dynamic stress drop. We define BS = 0 for points, where it was not possible to read DS, and points where the rupture onset is followed by an immediate stress increase (see Fig. 3c). Finally, we define a new quantity, stress delay $T_x$, as the delay between the rupture front and the peak stress arrivals. If the rupture front arrives coincides with the peak stress arrival, then $T_x = 0$. However, we have found some points along the fault where the peak stress precedes the rupture front arrival ($T_x < 0$). Taking into account simple friction (e.g. the slip weakening law), the fault would start slipping earlier at these points—immediately after the peak stress arrival, which does not agree with the prescribed rupture velocity. Hence, points $T_x < 0$ are very probably inconsistent with source dynamics. Although this opinion is oversimplified and more sophisticated constitutive laws (e.g. the rate and state friction) allow a broader class of stress time histories (even with non-zero $T_x$), we do not expect the $k$-squared model to imply such complex friction laws, as it is just a schematic kinematic model.

The spatial distributions of the dynamic parameters for Hisada’s SVF are depicted in Fig. 4. The white colour in the strength excess distribution indicates regions of negative or zero values (values below or equal to the initial stress), which is not physically consistent—these regions ruptured after a zero stress change or even after stress release. We recognize the strong effect of frequency filtering: the peak associated with the rupture front arrival is smeared by filtering (see Fig. 2). Since our filter is acausal, the stress drop may precede the rupture front. The problem disappears with increasing cut-off frequency. However, small regions of the negative strength excess remain close to the nucleation point. This is not surprising, as we start the rupture from a point, and not from a small but finite area, the latter being necessary in forward dynamic simulations (e.g. Andrews 1976). Bouchon (1997) found that the strength excess inversely correlates with the local rupture velocity. Since we assume a constant rupture velocity, the strength excess is partly correlated just with the static stress change. The breakdown stress drop is clearly correlated with the static stress drop and grows with increasing cut-off frequency. The white colour in the breakdown stress drop distribution indicates regions where the stress just rises in time after rupture front arrival (see Fig. 3c). Stress delay $T_x$ is non-zero in the vicinity of the nucleation point, as well as non-positive strength
excess regions. Other regions of non-zero $T_x$ disappear with increasing cut-off frequency, except the small areas near the border of the fault.

An example of stress time histories over the fault for Hisada’s SVF is shown in Fig. 5. The difference between the dynamic and the static stress drop distributions (DS-SS), the stress recovery, is displayed in the background. An interesting result is that the slip pulse propagates without strong recovery of stress along most of the fault, unlike the model of Heaton (1990). This was also found in the dynamic stress field studies of real earthquakes (Day et al. 1998). However, a closer look at Fig. 2 suggests that the amount of stress recovery would depend on the pulse width. Particularly in case of Dirac’s SVF, pulse width $L_0 \to 0$, and one can see strong stress recovery. Reminding the reader that $L_0 = v_r \tau_{max}$, we carried out the calculation for 9 combinations of $\tau_{max} \in \{0.5, 1, 2\}$ s and $v_r \in \{2.3, 2.6, 2.9\}$ km s$^{-1}$ with Hisada’s SVF, $f_{max} = 12$ Hz and the nucleation point at the left border of the fault. We found that the change in the pulse width only causes a change in the absolute values of (DS-SS). The average values of the stress recovery (DS-SS) are plotted against pulse width $L_0$ in Fig. 6, showing that (DS-SS) is inversely proportional to $L_0$. This is consistent with the analytic relations of Broberg (1978) and Freund (1979), valid for 2-D in-plane and anti-plane steady state pulses, respectively, which generally imply inverse proportionality of $L_0$ and DS (the constant of the proportionality depends slightly on the rupture velocity). All calculations resulted in the largest absolute values of (DS-SS) near the nucleation point (see Fig. 4). Even in case of narrow pulse $L_0 \simeq 1$ km, the stress recovery (DS-SS) is low (<2 MPa) farther away than $L/2$ from nucleation point.

The concept of the slip pulse was introduced to the $k$-squared model by Bernard et al. (1996) just by assumption. Slip pulses are
Dynamic - static stress drop, $v_r=2.6\text{km/s}$, $a=0.5$, $\tau_{\text{max}}=1\text{s}$, Hisada SVF, $f_{\text{max}}=12\text{Hz}$

Figure 5. An example of stress time histories over the fault for Hisada’s slip velocity function and cut-off frequency $f_{\text{max}}=12\text{Hz}$. We show just the 3.12 s window of every sixth time history in both directions. The difference between the dynamic and the static stress drop distributions (stress recovery) is displayed in the background. The nucleation point is indicated by an asterisk.
usually observed in kinematic models of past earthquakes (Heaton 1990), however, their origin is not clear. Two possible mechanisms were presented to explain the existence of slip pulses (for a detailed discussion refer to Ben-Zion 2001). One explanation is a large dynamic variation of the frictional force along the fault, which can be caused, for example, by a strong velocity dependence of the friction coefficient (e.g. Heaton 1990) or by the variation of the normal stress along the fault separating two materials with different properties—the so-called wrinkle pulse (e.g. Andrews & Ben-Zion 1997). Another explanation is that the slip pulse is just the result of fault heterogeneities, which was demonstrated not only in theoretical forward dynamic models (e.g. Das & Kostrov 1988), but also in models of particular earthquakes (Beroza & Mikumo 1996; Day et al. 1998). The latter explanation is consistent with the $k$-squared model with broader pulses ($L_0 > L/5$), as we do not observe stress recovery after dynamic weakening. The slip pulse could then be substantiated by the natural spatial heterogeneity of the $k$-squared model. The limit $L_0 > L/5$ confirms the value proposed by Bernard et al. (1996) for a crack like behaviour of “broad-pulse” slip model. In case of narrow pulses ($L_0 < L/5$) stress recovery is significant close to the nucleation point and, therefore, the slip pulse cannot be explained merely by fault heterogeneity.

In earthquake dynamics, the source process is controlled by the stress state surrounding the fault and the constitutive equations describing the relationships among the kinematic, mechanical, thermal or even chemical field variables along the fault. One of the simplest and most widely used constitutive equations is the slip weakening (SW) law introduced to earthquake source studies by Ida (1972). It relates shear traction along the fault only to the fault slip. In particular, a slip is zero until shear stress reaches a critical value—the yield stress. Once the yield stress is reached, the slip starts to grow while the shear stress decreases. After the slip exceeds the critical displacement $D_c$, the shear stress no longer decreases and remains constant at the so-called dynamic stress till the end of the rupture process. Although other constitutive laws exist, for example, the rate and state dependent friction law derived from laboratory rock friction measurements (Dieterich 1979; Ruina 1983; Marone 1998) and explaining even pre- and post-seismic phenomena, the SW law has been used successfully to interpret ground motions with dynamic models (e.g. Beroza & Mikumo 1996; Olsen et al. 1997; Guatn&er & Spudich 2000; Peyrat & Olsen 2004). Moreover, the SW behaviour was found to be a feature of the rate and state dependent friction law itself (Cocco et al. 2004, and references therein) and also in the dynamic stress field of the kinematic models derived by the inversion of earthquake waveforms (Ide & Takeo 1997). Since SW is an important feature of the dynamic stress change during earthquake rupture, we analysed the stress slip curves for the $k$-squared model. We fitted stress slip curves using the linear SW relation (the stress falls linearly with the slip), arriving at the optimal value of $D_c$. Particularly, we fixed both the strength excess and dynamic stress drop and performed a direct search in the interval $(0, D_{\text{tot}})$ for optimal $D_c$ (in the sense of the L2 norm), $D_{\text{tot}}$ is the final slip. An example of such optimal solution over the fault is in Fig. 7 and in more detail for eight selected points in Fig. 8. SW dominates in regions of positive static stress drop, where the variance reduction of the optimal solution exceeds 90 per cent. Regions of negative static stress drop and positive breakdown stress drop exhibit slip weakening or a combination of slip weakening and slip hardening. The SW fit is generally worse (variance reduction up to 50 per cent) in these regions. The SW fit has no meaning in regions of zero breakdown stress drop. The optimal values of $D_c/D_{\text{tot}}$ for three values of $f_{\text{max}}$ are plotted in Fig. 9. We show $D_{\text{tot}}/D_{\text{tot}}$ just for areas of positive static stress drop, as there is a high reduction variance of the slip weakening fit (over 90 per cent) for the three values of $f_{\text{max}}$ and thus the results are comparable. One can see that the values of $D_c/D_{\text{tot}}$ depend on the cut-off frequency—the lower $f_{\text{max}}$, the closer $D_c$ to $D_{\text{tot}}$. Generally, $D_c$ follows $D_{\text{tot}}$, but $D_{\text{tot}}$ slowly varies along the fault from 20 to 40 per cent, to 60 per cent to 100 per cent for $f_{\text{max}}$ = 12, 6 and 3 Hz, respectively. Total slip $D_{\text{tot}}$ versus $D_c$ for all points within positive stress drop areas is plotted in Fig. 10. The effect of filtering is clear, a lower $f_{\text{max}}$ pushes the values of $D_c$ closer to $D_{\text{tot}}$. Note the points $D_c = D_{\text{tot}}$. The scaling becomes less apparent for $f_{\text{max}}$ = 12 Hz, but generally still holds. Both the green and red lines denote maxima of the $D_c/D_{\text{tot}}$ frequency–magnitude distributions shown later in Fig. 12. The scaling of $D_c$ with the final slip was found by Mikumo et al. (2003), Zhang et al. (2003) and Tinti et al. (2005) for several earthquakes. However, Spudich & Guatniteri (2004) pointed out that this could be caused by the limited bandwidth of the kinematic inversions. Our results are similar to that of Zhang’s results for $f_{\text{max}} = 6$ and 12 Hz and Tinti’s results for $f_{\text{max}}$ = 3 Hz. On the other hand Mikumo presented lower values of $D_c/D_{\text{tot}}$ from the interval (0.27–0.52) for Tottori earthquake. Though, it has been already pointed by Mikumo et al. themselves, that their $D_c$ estimation method fails near strong barriers—exactly at the regions where we find $D_c/D_{\text{tot}} \approx 1$. Actually, the $D_c$ estimates for earthquakes are quite peculiar (Guatniteri & Spudich 2000) and it seems that recent kinematic source inversions were unable to resolve $D_c$ correctly due to the limited bandwidth of the data (Spudich & Guatniteri 2004; Piatanesi et al. 2004). Therefore, it is difficult to compare the $D_c$ from observational studies with the $D_c$ we obtained for the theoretical $k$-squared model in this study.

### 6 A PARAMETRIC STUDY

As the $k$-squared model with $k$-dependent rise time has a relatively high number of free model parameters (length and width of the fault, asperity slip contrast, slip roughness—represented by $K_c$, stochastic slip distribution, number and positions of asperities, position of nucleation point, rupture velocity, maximum rise time, coefficient $a$, and SVF type) it represents a large set of source models. We have
Stress vs. slip, $v_r=2.6\text{km/s}$, $a=0.5$, $\tau_{\text{max}}=1.\text{s}$, Hisada SVF, $f_{\text{max}}=12\text{Hz}$

Figure 7. Stress-slip diagrams for Hisada’s slip velocity function (black) with corresponding linear slip weakening fit (red). We show every sixth stress-slip diagram in both directions. The zero stress level is marked by a short dash. The nucleation point is indicated by an asterisk. The stress-slip curves denoted by A–H are shown in Fig. 8.
Figure 8. An example of stress-slip curves (black) and corresponding linear slip weakening fit (red) at eight points along the fault denoted in Fig. 7 by A–H, respectively. A worse fit in the negative static but positive breakdown stress drop areas is presented in the bottom row (E–H).

Figure 9. The effect of filtering on $D_c/D_{\text{tot}}$ (ratio of critical and total slip) distributions for Hisada’s slip velocity function. Three cut-off frequencies are considered ($f_{\text{max}} = 3, 6$ and 12 Hz). The nucleation point is indicated by an asterisk.

Figure 10. Total slip $D_{\text{tot}}$ versus critical slip $D_c$ at points within positive stress drop areas and for three filter cut-off frequencies ($f_{\text{max}} = 3, 6$ and 12 Hz). The red solid line indicates 1:1 ratio. The green dashed line indicates a maximum of the $D_c/D_{\text{tot}}$ frequency–magnitude distribution (see later Fig. 12).

performed a parametric study which covers some sections of this model space to explore its projection into the space of dynamic parameters, and to possibly find some restrictions on the $k$-squared model originating from source dynamics. The scope of the parametric study is limited. We have focused on the combination of low-wavenumber deterministic and high-wavenumber stochastic slip model of $M_w = 5.9$ earthquake. The low-wavenumber model (fault and total asperities area, asperity slip contrast, number of asperities—in this case just one deterministic asperity) is based on the empirical scaling relations of Somerville et al. (1999). We have varied the rupture velocity, maximum rise time, nucleation point position, SVF type, stochastic slip distribution and asperity position. Each of these were studied separately, fixing the other parameters at the reference values. The model, which was analysed in the previous sections (Hisada’s SVF, $v_r = 2.6$ km s$^{-1}$, $a = 0.5$, $\tau_{\text{max}} = 1$ s, $f_{\text{max}} = 12$ Hz) served as a reference model. The re-
Stress field of the $k$-squared model

Figure 11. Strength excess, breakdown stress drop, $D_c/D_{\text{tot}}$ (ratio of critical and total slip) distributions and stress delay $T_x$ for 14 different sets of $k$-squared model input parameters. (a) our reference model (Hisada’s slip velocity function, nucleation point at left border of the fault, rupture velocity 2.6 km s$^{-1}$); (b)–(d) three different values of the rupture velocity (2.3, 2.9, 3.18 km s$^{-1}$); (e)–(f) two different values of maximum rise time; (g)–(h) two different positions of the nucleation point (centre, right border of the fault); (i)–(j) two different types of slip velocity functions (Brune, Boxcar); (k) a different $k$-squared slip distribution—$K_c = 0.5$ slip distribution from Fig. 1(a); (l) a different stochastic realization of $K_c = 0.5$ slip distribution; (m) a different $k$-squared slip distribution—$K_c = 0.75$ slip distribution from Fig. 1(a); (n) a different stochastic realization of $K_c = 0.75$ slip distribution; (o) $K_c = 0.5$ slip distribution with a different position of the asperity.

Results of the parametric study are presented in Fig. 11. $D_c/D_{\text{tot}}$ is plotted just for areas of positive static stress drop, as the reduction variance of the slip weakening fit is 90 ± 5 per cent for all models in these areas, except for the case of $v_r = 3.18$ km s$^{-1}$ (65 per cent). Thus $D_c/D_{\text{tot}}$ should be comparable in the different models. Further, it was difficult to represent $D_c/D_{\text{tot}}$ by a single value (e.g. by the average value), since its frequency–magnitude distribution has 2 local maxima (at $D_c/D_{\text{tot}} = 1$ and $D_c/D_{\text{tot}} \approx 0.5$), hence we plot frequency–magnitude distribution for all models in Fig. 12.

6.1 Rupture velocity
We tested 3 sub-Rayleigh rupture velocities 2.3 km s$^{-1}$, 2.9 km s$^{-1}$ and 3.18 km s$^{-1}$ (Figs 11b–d). The last one is the Rayleigh velocity.
The strength excess (SE) distribution is very sensitive to rupture velocity \( v_r \). Specifically, the SE values decrease almost linearly with \( v_r \) (the average positive SE values decrease linearly from 5 MPa for \( v_r = 2.3 \text{ km s}^{-1} \) to 1.8 MPa for \( v_r = 3.18 \text{ km s}^{-1} \)). The spatial pattern of the SE distribution follows the SE reference solution (Fig. 11a). The negative SE values covering less than 5 per cent of the fault area close to the nucleation point are present for all the tested \( v_r \). When \( v_r \) reaches the Rayleigh velocity, negative SE values appear even outside the rupture nucleation region and cover about 30 per cent of the fault.

The dynamic stress drop (DS) did not change with \( v_r \), hence the breakdown stress drop (BS) follows the SE distribution, as BS = DS + SE (the average BS values decrease linearly from 8.9 MPa for \( v_r = 2.3 \text{ km s}^{-1} \) to 5.4 MPa for \( v_r = 3.18 \text{ km s}^{-1} \)). The \( D_c/D_{tot} \) distribution does not change much with \( v_r \). The \( T_x \) distribution depends on \( v_r \). Non-zero \( T_x \) covers less than 5 per cent of the fault area close to the nucleation point for \( v_r = 2.3 \text{ km s}^{-1} \), 2.6 \text{ km s}^{-1}. \( T_x \) becomes non-zero outside the rupture nucleation region with \( v_r \) approaching the Rayleigh velocity (non-zero \( T_x \) at 21 per cent and 67 per cent of the fault area, for \( v_r = 2.9 \) and 3.18 \text{ km s}^{-1}, respectively).

### 6.2 Maximum rise time

We then tested two values of the maximum rise time — 0.5 and 2 s (Figs 11e–f). SE increased with shorter rise time (5 MPa average...
Figure 11 – Continued

SE for \(\tau_{\max} = 0.5\) s and, vice versa, SE decreased with longer rise time (2.9 MPa average SE for \(\tau_{\max} = 2.9\) s). The change in BS is not hidden just in SE as in the case of variable \(v_r\). We obtained 9.3 and 6.1 MPa average BS for shorter and longer \(\tau_{\max}\), respectively. Also there is a clear dependence of \(D_c/D_{\text{tot}}\) on \(\tau_{\max}\): the longer \(\tau_{\max}\), the smaller \(D_c/D_{\text{tot}}\). Also, the non-zero \(T_x\) area (10 per cent of the fault area) is larger for longer \(\tau_{\max}\).

6.3 Nucleation point position

The change in the position of the nucleation point (Figs 11g–h) leads to clear changes in the spatial patterns of the SE, BD, \(D_c/D_{\text{tot}}\) and \(T_x\) distributions. However, the average values of both SE and BS are very close to the reference solution (average SE = 3.8, 3.6, 3.8 MPa and average BS = 7.4, 7.4, 7.5 MPa, for a rupture starting from the left border, centre and right border of the fault, respectively). \(T_x\) is zero almost for the whole fault (less than 1 per cent of the fault area) in case of the rupture nucleating from the centre of the fault and non-zero along less than 5 per cent of the fault for the other two nucleation points.

6.4 Slip velocity function shape

Further, we tested Boxcar and Brune’s slip velocity functions (Figs 11i–j). The SE (1.8 MPa average), BS (6.4 MPa average) and \(T_x\) distributions for the Boxcar SVF resemble the results for \(\tau_{\max} = 2.9\) s (see Fig. 11f), which can be explained by a shorter ‘efficient’ duration of Hisada’s SVF (e.g. fig. 6 in Gallovic & Brokesova 2004). The distributions of SE (3.6 MPa average) and BS (7.8 MPa average) in the case of Brune’s SVF are close to the results for Hisada’s SVF. Non-zero \(T_x\) covers less than 10 per cent of the fault and appears even outside the rupture nucleation region. The \(D_c/D_{\text{tot}}\) values tend to 1 over most of the fault in the case of the Boxcar SVF. The \(D_c/D_{\text{tot}}\) values are also higher for Brune’s SVF than for Hisada’s SVF.
6.5 Static slip distribution

Finally, we performed the analysis for other static slip distributions (Figs 11 k–o), particularly for the both $K_c = 0.5$ and $K_c = 0.75$ slip distributions from Fig. 1(a), two different stochastic distributions of these two and a different position of the asperity. The average SE (3.9, 3.9 MPa) and BS (7.6, 7.7 MPa) values are very close to the reference solution (3.8 and 7.4 MPa, respectively) for the two stochastic realizations of $K_c = 0.5$ slip distribution. However, the spatial patterns of the SE and BS distributions differ in details from the reference solution. The differences are mainly due to the differences in the static stress change distributions (see Fig. 1b). The $D_1/D_{tot}$ and $T_s$ (non-zero along less than 5 per cent of the fault) distributions are close to the reference solution. The average SE (4.3, 4.4 MPa) and BS (9.4, 9.8 MPa) values are higher for the two stochastic realizations of $K_c = 0.75$ slip distribution. Non-zero $T_s$ covers less than 10 per cent of the fault and appears even outside the rupture nucleation region. The frequency–magnitude distribution of $D_1/D_{tot}$ is similar to the reference solution, however, as the areas of negative stress drop are larger and static stress change is more complicated (Fig. 1b) the $D_1/D_{tot}$ spatial distribution is also more complicated. Zero BS area also increases considerably with $K_c = 0.75$. The average SE (3.8 MPa) and BS (7.3 MPa) values are very close to the reference solution for the different position of the asperity. The spatial patterns of the SE and BS distributions differ from the reference solution, it follows the static stress change distribution. The $D_1/D_{tot}$ and $T_s$ (non-zero along less than 5 per cent of the fault) distributions are also close to the reference solution.

We emphasize that the results do not depend on a single stochastic realization, except for the changing of the pattern of small scale features. The results depend mostly on the position of the asperity. Even in the case of rougher fault ($K_c = 0.75$), shifting more power to higher wavenumbers, the influence of asperity is still present – the bend of negative stress drops enclosing the asperity. The $D_1/D_{tot}$ frequency–magnitude distributions also change very little with the different slip distributions.

7 DISCUSSION AND CONCLUSIONS

The purpose of this paper was to investigate both the static and dynamic stress change due to a theoretical kinematic source model: the $k$-squared source model with asperities. We found that the stress field of the $k$-squared slip distribution and the slip velocity function with $k$-dependent rise time is free of singularities. Let us compare the results with the findings of Madariaga (1978), who studied Haskell’s fault model. Contrary to Haskell’s model with an infinite average static stress drop, the $k$-squared model has a finite average static stress drop since the slip vanishes smoothly at the border of the fault, so that the static stress change is bounded. Further, the $k$-squared model with $k$-dependent rise time generates stress time histories which are also bounded as the slip velocity functions have finite rise times. On the other hand, stress time histories generated by Haskell’s model with instantaneous slip contain singularities associated with $S$ wave and rupture front arrivals. Thus, the $k$-squared model with $k$-dependent rise time is not in such clear contradiction to earthquake source dynamics as Haskell’s model.

A detailed analysis was carried out to compare the stress state along the fault with the constitutive relations used in earthquake source dynamics. Attention was paid to the stress recovery associated with the slip pulse, the slip pulse being an ingredient of the $k$-squared model with $k$-dependent rise time. We found the stress recovery to be close to the nucleation point for narrow pulses ($L_0 < L/5$). We point out that the amount of stress recovery depends on the rupture velocity and rise time, that is, on the pulse width. Further, we determined the strength excess, breakdown stress drop and dynamic stress drop distributions. We found that it was also possible to fit stress slip curves to a simple linear slip weakening law, obtaining $D_s$ with high variance reduction (~90 per cent) in the positive static stress drop areas. Stress delay $T_s$, a new parameter introduced in this paper, agrees with the yielding criteria of the simple slip weakening friction law (characterized only by yield stress, dynamic stress and critical slip weakening distance $D_s$).

The stress time histories due to the $k$-squared model are very similar to the stress time histories due to the kinematic models of real earthquakes (see figs 4 and 5 in Day et al. 1998). The values of the strength excess and dynamic stress drop are in tens of MPa as for real earthquakes as found, for example, by Bouchon (1997), Piatanesi et al. (2004). The average value of $D_1/D_{tot}$ in the positive stress drop area is $0.65 \pm 0.05$ (except for the cases of Brune’s SVF, Boxcar SVF and $\tau_{max} = 0.5$ s) which is similar to the value 0.63 found by Zhang et al. (2003) for the 1999 Chi-Chi earthquake. $D_1/D_{tot}$ frequency–magnitude distributions present two maxima. One maximum lays around 50 per cent and the other at 100 per cent. $D_s$ is found at the edges of the both asperity and fault.

The parametric study helped us to reject or adopt some values of the free parameters of the $k$-squared model, taking into account the simple slip weakening friction law. Particularly, rupture velocities close to the Rayleigh velocity $v^R$ lead to a worse linear slip weakening fit (65 per cent variance reduction for $v_s = v^R$) and large areas of both negative strength excess and non-zero $T_s$. Thus we conclude that rupture velocities $0.9 v^R$ to $v^R$ are not suitable for the $k$-squared model with $k$-dependent rise time. Super shear rupture velocities were not studied in this paper. The constant rupture velocity in the $k$-squared model seems to be the most problematic constraint. The phenomenon of constant rupture velocity is not present even in simple forward dynamic problems of spontaneous rupture propagation. However, a constant rupture velocity can be modelled by heterogeneous frictional parameters (e.g. by the strength excess and breakdown stress drop distributions obtained in our study). Nevertheless, a constant rupture velocity is very unlikely, and the $k$-squared model should be refined in this sense to become more realistic.

Non-zero $T_s$ vanishes with short maximum rise time ($\tau_{max} = 0.5$ s), however, stress recovery after a pulse passage increases and $D_s$ becomes closer to $D_{tot}$. On the other hand, longer rise time ($\tau_{max} = 2$ s) leads to lower $D_s$, negligible stress recovery, but to larger areas of non-zero $T_s$. Thus we conclude that the maximum rise time $\tau_{max} = 1$ s is optimal for our fault dimensions and elastic parameters. It is necessary to extend the parametric study to generalize this conclusion. Mai et al. (2005) found more probable ruptures nucleating from regions close to asperities and not from zero slip areas. Our results partially agree with these findings. The nucleation point in the centre of the fault (non-zero slip, asperity border) leads to smallest non-zero $T_s$ area, however, areas of non-zero $T_s$ cover less than 5 per cent of the fault for the two other positions of nucleation points (fault border—zero slip). Thus, we do not prefer any position of the studied nucleation points. The area of non-zero $T_s$ is smallest for Hisada’s slip velocity function. Also $D_s$ is shortest for Hisada’s SVF. Hence, employing Hisada’s SVF is more consistent with applications of the slip weakening law in earthquake source dynamics than Boxcar, Brune’s and Dirac’s SVF. We attribute it to its similarity with the Kostrov function, which is an analytical solution of the forward dynamic problem. We conclude that nine $k$-squared models with dynamic parameters plotted in Figs 11(a), (b), (g), (h),
ACKNOWLEDGMENTS

We would like to thank Franta Gallovic for his slip and slip velocity generators, Paul Spudich, Luis Dalguer, Martin Mai, Yehuda Ben-Zion and anonymous reviewer for constructive comments and helpful reviews. The research was supported by research project of the Grant Agency of the Charles University (279/2006/B-GEO/MFF), Grant Agency of the Czech Republic (205/07/0502), and 3HAZ- Grant Agency of the Charles University (279/2006/B-GEO/MFF), Zion and anonymous reviewer for constructive comments and help-generators, Paul Spudich, Luis Dalguer, Martin Mai, Yehuda Ben-Zion and anonymous reviewer for constructive comments and helpful reviews. The research was supported by research project of the Grant Agency of the Charles University (279/2006/B-GEO/MFF), Grant Agency of the Czech Republic (205/07/0502), and 3HAZ-CORINTH - STREP of the EC 6th Framework Program (GOCE-4043).

REFERENCES


