Appendix to the paper "Heterogeneous Coulomb stress perturbation during earthquake cycles in a 3D rate-and-state fault model" by F. Gallovič

Derivation of the kernel accounting for static part of the dynamic elastic interactions among fault cells

Let the fault be on the $y = 0$ plane in a Cartesian coordinate system, and let the slip be in the x direction. In 3D infinite homogeneous isotropic elastic medium (characterized by Lamé's parameters G and λ) the static on-fault shear-stress response $\tau = \tau_{xy}$ to strike-slip motion defined by the static slip distribution $\Delta u = \Delta u_x$ is given by the representation theorem (Andrews, 1974)

$$\tau(x', z') = \frac{G}{4\pi} \iint \left[\frac{2(\lambda + G)}{\lambda + 2G} \frac{1}{r(x', z', x, z)} \frac{\partial^2 \Delta u(x, z)}{\partial x^2} + \frac{1}{r(x', z', x, z)} \frac{\partial^2 \Delta u(x, z)}{\partial z^2} \right] dxdz, \quad (1)$$

where x, x', z and z' denote positions on the fault and distance r has the form

$$r(x', z', x, z) = \left[(x - x')^2 + (z - z')^2 \right]^{1/2}. \quad (2)$$

Considering that the slip is constant over grid cells of size $\Delta x \times \Delta z$, we can write using the 1-D Heaviside function $H(\xi)$

$$\Delta u(x, z) = \sum_{mn} d_{mn} \left[H(x - (m - 1/2)\Delta x) - H(x - (m + 1/2)\Delta x) \right] \times \left[H(z - (n - 1/2)\Delta z) - H(z - (n + 1/2)\Delta z) \right]. \quad (3)$$

The second spatial derivative in Eq. (1) of slip distribution (3) with respect to x is

$$\frac{\partial^2 \Delta u(x, z)}{\partial x^2} = \sum_{mn} d_{mn} \left[\delta_{,x}(x - (m - 1/2)\Delta x) - \delta_{,x}(x - (m + 1/2)\Delta x) \right] \times \left[H(z - (n - 1/2)\Delta z) - H(z - (n + 1/2)\Delta z) \right], \quad (4)$$

where $\delta_{,x} = \partial\delta/\partial x$ is a derivative of the Dirac δ-function. Analogously, the second spatial derivative with respect to z reads

$$\frac{\partial^2 \Delta u(x, z)}{\partial z^2} = \sum_{mn} d_{mn} \left[H(x - (m - 1/2)\Delta x) - H(x - (m + 1/2)\Delta x) \right] \times \left[\delta_{,z}(z - (n - 1/2)\Delta z) - \delta_{,z}(z - (n + 1/2)\Delta z) \right] \quad (5)$$

with $\delta_{,z} = \partial\delta/\partial z$.

Inserting (4) and (5) into (1) we are about to integrate four terms where the first one is

$$T_1^{mn}(x', z') = \iint \frac{1}{r(x', z', x, z)} \delta_{,x}(x - (m - 1/2)\Delta x) \times \left[H(z - (n - 1/2)\Delta z) - H(z - (n + 1/2)\Delta z) \right] dxdz. \quad (6)$$

Due to the Heaviside functions the integration interval along z is bounded between $(n-1/2)\Delta z$ and $(n+1/2)\Delta z$. Since for general function $f(x)$ we can use the identity $\int \delta_{,x}(x-a)f(x)dx = -f_{,x}(a)$, we can integrate (6) with respect to x and obtain after simple algebra

$$T_1^{mn}(x', z') = \int_{(n-1/2)\Delta z}^{(n+1/2)\Delta z} \frac{(m-1/2)\Delta x - x'}{[((m-1/2)\Delta x - x')^2 + (z-z')^2]^{3/2}} dz. \tag{7}$$

Finally, we integrate (7) with respect to z and evaluate the result at point ($x' = k\Delta x$; $z' = l\Delta z$)

$$T_1^{mn}(k\Delta x, l\Delta z) = \frac{1}{I_{m-k}^-}\left(\frac{J_{n-l}^+}{\left[(I_{m-k}^-)^2 + (J_{n-l}^+)^2\right]^{1/2}} - \frac{J_{n-l}^-}{\left[(I_{m-k}^-)^2 + (J_{n-l}^-)^2\right]^{1/2}}\right) = T_1^{m-k,n-l} \tag{8}$$

where

$$I_{m-k}^- = (m-k-1/2)\Delta x \tag{9}$$
$$J_{n-l}^- = (n-l-1/2)\Delta z \tag{10}$$
$$J_{n-l}^+ = (n-l+1/2)\Delta z \tag{11}$$
$$\tag{12}$$

Using analogous definition

$$I_{m-k}^+ = (m-k+1/2)\Delta x, \tag{13}$$

we can analogously derive the remaining three terms:

$$T_2^{mn}(k\Delta x, l\Delta z) = \frac{1}{I_{m-k}^+}\left(\frac{J_{n-l}^+}{\left[(I_{m-k}^+)^2 + (J_{n-l}^+)^2\right]^{1/2}} - \frac{J_{n-l}^-}{\left[(I_{m-k}^+)^2 + (J_{n-l}^-)^2\right]^{1/2}}\right) = T_2^{m-k,n-l} \tag{14}$$

$$T_3^{mn}(k\Delta x, l\Delta z) = \frac{1}{J_{n-l}^-}\left(\frac{I_{m-k}^+}{\left[(I_{m-k}^+)^2 + (J_{n-l}^-)^2\right]^{1/2}} - \frac{I_{m-k}^-}{\left[(I_{m-k}^-)^2 + (J_{n-l}^-)^2\right]^{1/2}}\right) = T_3^{m-k,n-l} \tag{15}$$

$$T_4^{mn}(k\Delta x, l\Delta z) = \frac{1}{J_{n-l}^+}\left(\frac{I_{m-k}^+}{\left[(I_{m-k}^+)^2 + (J_{n-l}^+)^2\right]^{1/2}} - \frac{I_{m-k}^-}{\left[(I_{m-k}^-)^2 + (J_{n-l}^+)^2\right]^{1/2}}\right) = T_4^{m-k,n-l}. \tag{16}$$

The traction (1) at discrete points $x' = k\Delta x$, $z' = l\Delta z$ due to the slip distribution (3) can be then written as a discrete convolution

$$\tau_{kl} = \sum_{mn} K_{m-k,n-l} d_{mn}, \tag{17}$$

where kernel $K_{m-k,n-l}$ reads

$$K_{m-k,n-l} = \frac{G}{4\pi}\left[\frac{2(\lambda+G)}{\lambda+2G}\left(T_1^{m-k,n-l} - T_2^{m-k,n-l}\right) + T_3^{m-k,n-l} - T_4^{m-k,n-l}\right]. \tag{18}$$

Renaming subscripts kl and mn to i and j, respectively, we arrive to the expression used in Eq. (5) in the paper.