Moment Tensor Resolvability: Application to Southwest Iberia

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Abstract We present a method to assess the uncertainty of earthquake focal mechanisms based on the standard theory of linear inverse problems. We compute the uncertainty of the moment tensor, $\mathbf{M}$, then map it into uncertainties of the strike, dip, and rake. The inputs are: source and station locations, crustal model, frequency band of interest, and an estimate of data error. The output is a six-dimensional (6D) error ellipsoid, which shows the uncertainty of the individual parameters of $\mathbf{M}$. We focus on the double-couple (DC) part of $\mathbf{M}$. The method is applicable both with and without waveforms. The latter is particularly useful for network design. As an example we present maps of DC resolvability for earthquakes in southwest Europe, computed without waveforms. We find that the resolvability depends critically on frequency range and source depth. Shallow DC sources (10 km) are theoretically better resolved than deeper sources (40 km and 60 km). The DC resolvability of a 40-km-deep event improves considerably when the Portuguese network is supplemented by stations in Spain and Morocco. The DC resolvability can be further improved by using a few ocean bottom seismometer (OBS) stations or a dense land network. A dense land network is able to resolve $\mathbf{M}$ well in spite of the large azimuthal gap, which spans $\sim 200^\circ$.

The theoretical resolution analysis also explains the success of single-station inversions when using a broad frequency range, as exemplified by an application using waveforms of a $M_w$ 6 earthquake offshore Iberia.

Introduction

Earthquake models can be divided into two broad categories: point sources and finite sources. Finite-fault models contain information about the spatial evolution of slip on the fault. Point-source models simplify the earthquake as slip that occurs on a single point in space. Point-source models are useful to describe the geometry of faulting as well as nonshear components of the source, even for small- and moderate-size earthquakes. These parameters are quite useful, for example, to understand the tectonics of a given region. The most general and widely used description of a seismic point source is the moment tensor (MT), a second-rank, three-by-three, symmetric tensor (Gilbert, 1971; Aki and Richards, 2002). The moment tensor, $\mathbf{M}$, describes any source of deformation in an elastic medium in terms of force-couples. The MT of shear/tectonic earthquakes corresponds to a simple double-couple (DC) of forces. In this case, $\mathbf{M}$ contains information both about the magnitude of the earthquake and the geometry of faulting. The strike, dip, and rake angles can be derived from $\mathbf{M}$.

At present, seismologists use ground-motion waveforms to infer $\mathbf{M}$ on a routine basis. The problem of studying the seismic source given observed ground motion is an inverse problem (the corresponding direct problem is to compute the ground motion caused by a seismic source). The result of any inverse problem has limited reliability. The inversion of $\mathbf{M}$ is now common practice; however, the errors of $\mathbf{M}$ are rarely reported. The assessment of the reliability of $\mathbf{M}$ is particularly important when the dataset is less than perfect, for example, when only a few stations are available, when a large azimuthal gap exists, or when the signal-to-noise (S/N) ratio is low in the frequency band of interest. Why are the uncertainties of $\mathbf{M}$ not routinely reported? We can identify two main reasons: (1) It is relatively easy to compute error estimates for the individual components of $\mathbf{M}$, because they are linearly related to the waveforms (for a fixed depth and origin time). However, the more intuitive faulting parameters of strike, dip, and rake angles are not linearly related to the waveforms, so their uncertainties are more difficult to estimate. (2) The evaluation of the absolute errors of $\mathbf{M}$ requires an assessment of data error, which in principle cannot be obtained. A third reason may be suggested: Users of $\mathbf{M}$ (or strike, dip, and rake) are often more interested in a final result than in a lengthy/extensive discussion of error. However, using the results of an inversion without considering its errors is dangerous: A solution to an inverse problem can always be found; however, the solution will be meaningless if the errors are too large. This paper focuses on the uncertainty of $\mathbf{M}$. We address both issues stated previously: (1) we show how to compute the errors of $\mathbf{M}$ and how to translate them into errors of strike, dip and rake; (2) we also discuss
data errors, point ways to estimate their magnitude, and indicate applications for which an actual assessment of data error is not needed. Finally, the method that we present is simple and fast; thus, it can be implemented for routine inversions of $\mathbf{M}$.

The resolution of the moment tensors retrieved from waveform inversion has concerned seismologists for a long time (e.g., Riedesel and Jordan, 1989; Vasco, 1990). Nevertheless, the issue remains a topic of active research. Riedesel and Jordan (1989) showed how to use the covariance of $\mathbf{M}$, which is a fourth-rank tensor, to graphically visualize the uncertainty of moment tensors. However, their method is based on perturbation theory; hence, it is only applicable to small parameter variations (Vasco, 1990). Vasco (1990) studied the errors in $\mathbf{M}$ arising from noise in data and imperfect Green’s functions. He proposed a method to map data errors into bounds of three invariants of $\mathbf{M}$: the trace (which corresponds to the isotropic component of the source), the determinant, and the sum of the diagonal minors of the determinant. More recently, Yagi and Fukahata (2011) proposed a finite-fault inversion algorithm that takes into account the uncertainty of the Green’s functions. They computed the covariance matrix from the misfit between observed and synthetic waveforms. Then they rigorously included the covariance in the inverse problem to iteratively improve the estimate of the model parameters. Vavryčuk (2007) and Bukchin et al. (2010) showed that trade-offs between individual components of the moment tensor impose limits on the resolvability of $\mathbf{M}$, even for error-free data and Green’s functions (GFs). Such trade-offs are caused by symmetries in the seismic wave field and/or source-station configuration. In such situations, different earthquakes (as defined by $\mathbf{M}$) generate indistinct waveforms at the stations where data is recorded.

A popular approach to assess the robustness of $\mathbf{M}$ is to find a family of acceptable solutions rather than just one best-fitting solution. In this case, the family itself is used to define the confidence intervals of the model parameters. For example, Šílený (1998) generated families of acceptable $\mathbf{M}$s using a genetic algorithm. Some authors repeat their inversions with data contaminated with artificial noise, where the amplitude of the noise can be as high as 100% of the data amplitude (Hingee et al., 2011). Others repeat the inversions in artificially perturbed crustal models (Šílený, 2004; Wéber, 2006; Vavryčuk, 2007). Ford et al. (2009) discussed inaccurate crustal models in a very practical way in terms of temporal shifts between observed and synthetic waveforms. Ford et al. (2010) proposed the network sensitivity solution (NSS), a way to assess the reliability of the non-DC components of $\mathbf{M}$. The NSS is obtained through a grid search over the whole space of model parameters. Misfits (or equivalently, variance reductions) are computed for all possible source models. The results are displayed using a source-type plot, which allows the easy identification of shear versus volumetric components of $\mathbf{M}$ (Hudson et al., 1989). All these methods provide families of acceptable solutions without specifying absolute confidence intervals. In fact, absolute confidence intervals can only be computed given an independent assessment of data error, which cannot be obtained. These studies show that defining a range of acceptable solutions is useful, even if just in a relative sense (i.e., without determining absolute probabilities for specific confidence levels).

In this paper we present a simple approach: We use the well-established theory of linear inverse problems to map data errors into errors of the model parameters (e.g., Menke, 1989; Press et al., 1992; Parker, 1994). Thus, we obtain an estimate of the uncertainty of $\mathbf{M}$. We then map the uncertainty of $\mathbf{M}$ into uncertainties of strike, dip, and rake. Our approach is similar to that of Vasco (1990), in that we numerically map data errors into errors of the model parameters. Another method that shares similarities with ours is the NSS of Ford et al. (2010). The NSS extensively explores the parameter space in order to image the uncertainty of $\mathbf{M}$. However, the NSS requires the repeated computation of synthetic ground motion and variance reduction (or misfit). Our method does not require the actual computation of synthetic ground motion; it is simply based on the analysis of the Green’s functions. Also, our method allows an easy assessment of the resolvability of pure DC sources. The uncertainty of pure DC sources is not visualized in the NSS, because all pure DC solutions are collapsed onto a single point of the source-type plot.

The uncertainty analysis presented in this paper uses the full moment tensor. However, we focus on the DC part of $\mathbf{M}$, leaving the resolvability of non-DC components for a separate study. The uncertainty of $\mathbf{M}$, as presented in this paper, is useful in two distinct situations: (1) waveforms are available, they have been inverted to find $\mathbf{M}$, and the resulting source parameters need error bars; (2) no waveforms are available, but we are still interested in the resolvability of a given focal mechanism. The latter is typical of network design or experiment planning. The tools that we present in this paper are simple and based on standard theory; nevertheless, they provide a fast and computationally cheap way to assess the resolvability of $\mathbf{M}$, both with and without data.

Examples of the assessment of DC uncertainty are shown for earthquakes in southwest Iberia. First, we study the resolvability of a reference earthquake located offshore. We then investigate how the resolvability of DC solutions varies with earthquake depth, frequency range, crustal structure, station configuration, and data error. We also present maps of resolvability for different network configurations. Finally, we show an application using waveforms, where we explain the success of single-station inversions when using a broad frequency range. Some of the questions that we answer are: Is it enough to use the Portuguese network in order to resolve DC parameters for offshore earthquakes? Or do we need to add stations from neighbor countries? Or do we even need to add ocean bottom seismometers (OBSs), thus improving the azimuthal coverage? This last question has practical implications, because the deployment of OBS stations is
more expensive and difficult than that of traditional land stations.

Method

In this section we describe the linear inverse problem of finding the moment tensor $\mathbf{M}$ given observed ground motion. We then show how to compute the resolvability of the inferred model parameters. We also indicate how these values can be translated into uncertainties of strike, dip, and rake. Intentionally, part of the theory is presented in the Appendix; both because the theory behind the computations is standard and because moving the theory to the Appendix allows a more intuitive interpretation of the concepts.

The Moment Tensor

Ground motion is linearly related to the moment tensor:

$$d_i(t) = \sum_{j=1}^{6} G_{ij}(t)m_j, \quad (1)$$

In equation (1), $d_i(t)$ are the displacement waveforms (each one recorded on a different station/component $i$), $G_{ij}(t)$ is the matrix of Green’s functions (which we compute based on crustal structure, source location, station location, and frequency range), and $m_j$ are the unknown parameters that fully determine $\mathbf{M}$. Suppose that we have $N$ waveforms to use in the inversion, such that $i = 1, \ldots, N$. The number of model parameters to determine is six, equal to the number of independent components of $\mathbf{M}$. Then $\mathbf{d}(t)$ is a column vector of size $N \times 1$, $\mathbf{G}(t)$ is a matrix $N \times 6$, and $\mathbf{m}$ is a column vector of size $6 \times 1$. The inversion is formally overdetermined when $N \gg 6$.

The six parameters of $m_j$ may be the actual moment tensor components. More conveniently, $m_j$ may also be the weights of elementary focal mechanisms (Kikuchi and Kanamori, 1991; Zahradník et al., 2005; Zahradník, Janský, and Plicka, 2008; Zahradník, Sokos, et al., 2008). In this case, the columns of $\mathbf{G}$ are the so-called elementary seismograms, which correspond to the six elementary moment tensors $\mathbf{M}_1, \ldots, \mathbf{M}_6$. The moment tensor $\mathbf{M}$ is constructed as a linear combination of $\mathbf{M}_j$:

$$\mathbf{M} = m_1 \mathbf{M}_1 + \ldots + m_6 \mathbf{M}_6. \quad (2)$$

In our analysis we assume that the time dependence of $\mathbf{M}$ is known (e.g., the source time function is a delta function). We further assume that the earthquake origin time and location are known. Finding $\mathbf{M}$, or solving equations (1) and (2), is then a linear problem.

Best Solution versus Range of Acceptable Solutions

The goal of the inverse problem is to find a set of model parameters ($m_j$) that generates synthetic ground motion identical to the data. The solution to this problem is found by minimizing the misfit between observed data ($d_i$) and synthetics ($s_i$). Synthetics are given by the right side of equation (1), such that $s = \mathbf{Gm}$. We can proceed in two ways: (1) we can compute a single best (minimum-misfit) solution, for example, through least-squares minimization (LSQ); or (2) we can find a set of acceptable solutions with misfits ranging from a minimum to a threshold value. Option (1) is attractive when we look for a single solution for a genuinely linear problem, such as the inversion of $\mathbf{M}$. However, we favor option (2) because it provides a distribution of acceptable parameters (e.g., strike, dip, and rake angles), or a distribution of the deviations from their optimal values. Option (2) allows us to learn about nonoptimal solutions that still match the data well.

Let us start with case (1): We want to find a single best solution. The least-squares procedure minimizes the squared misfit, $e^2$:

$$e^2 = \sum_i (d_i - s_i)^2. \quad (3)$$

The summation is performed over all stations, components, and times. In the Appendix we outline two different ways of obtaining the LSQ solution, using normal equations and singular value decomposition (SVD).

To define a class of acceptable solutions, we introduce a weighted misfit that takes into account the data variance, $\sigma_i^2$, corresponding to the standard deviation, $\sigma_i$, of data $i$:

$$\chi^2 = \sum_i \left(\frac{d_i - s_i}{\sigma_i}\right)^2. \quad (4)$$

Note that in a strict sense only statistical errors are taken into account in $\sigma_i$; that is, the errors that would vanish if we could repeat one observation enough times and then compute an average, for example, instrumental noise, sensor misalignment, clock drift, etc. Systematic errors, that is, those that would not disappear with repetition and averaging, such as errors in the Green’s functions, are not taken into account in $\sigma_i$. Let us assume that all the waveforms are affected by the same data variance, $\sigma_d^2$. (This assumption simplifies the mathematical derivation; an analogous derivation using different $\sigma_i$ for each data piece is straightforward.) In this case $\sigma_i = \sigma_d$ for all waveforms $i$, and equation (4) simplifies to

$$\chi^2 = \frac{\sum_i (d_i - s_i)^2}{\sigma_d^2}. \quad (5)$$

The set of model parameters that generates the lowest $\chi^2$ is the optimal solution, which we will name the $\chi^2_{\text{min}}$ solution. We define the class of acceptable solutions as those for which the weighted misfit has a maximum value of $\chi^2_{\text{th}}$, where the subscript th stands for threshold:

$$\chi^2_{\text{th}} = \chi^2_{\text{min}} + \Delta \chi^2. \quad (6)$$
\( \Delta \chi^2 \) is the tolerated increase in misfit with respect to the minimum-misfit solution. The larger \( \chi_{th}^2 \) is, and correspondingly \( \Delta \chi^2 \), the more solutions are considered acceptable. We can then define the family of acceptable solutions by setting a limiting value for \( \Delta \chi^2 \):

\[
\Delta \chi^2 = \chi_{th}^2 - \chi_{min}^2,
\]

\[
\Delta \chi^2 = \frac{e_{th}^2 - e_{min}^2}{\sigma_d^2},
\]

\[
e_{th}^2 = e_{min}^2 + \sigma_d^2 \Delta \chi^2. \tag{7}
\]

We can easily see that the solutions for which \( \Delta \chi^2 < 1 \) are those whose squared misfits \( (e_{th}^2) \) are larger than the squared minimum-misfit \( (e_{min}^2) \) by less than the data variance \( (\sigma_d^2) \). The number of solutions within a \( \Delta \chi^2 \) interval depends explicitly on data error. More solutions are considered acceptable when \( \sigma_d^2 \) is larger.

In actual moment-tensor inverse problems, it is common practice to maximize the variance reduction, \( VR \), instead of minimizing the misfit \( e^2 \):

\[
VR = 1 - \frac{\sum (d_i - s_i)^2}{\sum d_i^2}. \tag{8}
\]

One can then choose a threshold value for the variance reduction \( (VR_{th}) \), above which solutions are considered acceptable. The variance-reduction threshold \( VR_{th} \) can be rewritten using equations (5), (6), and (8) as

\[
VR_{th} - VR_{min} = \left( 1 - \chi_{th}^2 \frac{\sigma_d^2}{\sum d_i^2} \right) - \left( 1 - \chi_{min}^2 \frac{\sigma_d^2}{\sum d_i^2} \right)
\]

\[
= -(\chi_{th}^2 - \chi_{min}^2) \frac{\sigma_d^2}{\sum d_i^2} = -\sum_{i} \frac{\Delta \chi^2 \sigma_d^2}{d_i^2}, \tag{9}
\]

and

\[
VR_{th} = VR_{min} - \frac{\Delta \chi^2 \sigma_d^2}{\sum d_i^2}. \tag{10}
\]

Equations (9) and (10) demonstrate that choosing a set of acceptable solutions via prescription of a variance-reduction threshold, \( VR_{th} \), is in fact equivalent to assigning a value to the data variance, \( \sigma_d^2 \), and to the acceptable \( \Delta \chi^2 \). For example, accepting only the optimal solution (i.e., \( VR_{th} = VR_{min} \)) is equivalent to assuming that either: (1) there is a family of acceptable solutions \( (\Delta \chi^2 > 0) \), but the data is error-free \( (\sigma_d = 0) \); or (2) the data has errors \( (\sigma_d > 0) \), but we accept only the optimum solution \( (\Delta \chi^2 = 0) \). Equation (10) also shows that, for \( \Delta \chi^2 > 0 \), a good numerical value for the variance reduction (i.e., \( VR \) close to \( VR_{min} \)) is meaningless if \( \sigma_d^2 \) is underestimated.

In this subsection we introduced the concept of an ensemble of acceptable solutions based on data error and \( \Delta \chi^2 \). This concept is very natural if we have a dataset, and various solutions match the data almost as well as the optimal solution. Another important point that we must take into account in order to compute meaningful uncertainties of the model parameters is data error. Next we will focus on the determination of the ensemble of acceptable solutions according to the \( \chi^2 \) criteria.

**Resolvability of the Moment Tensor**

In **Best Solution versus Range of Acceptable Solutions** we explored a situation in which we had a dataset with data variance \( \sigma_d^2 \) and wanted to find a family of acceptable solutions of \( \bf{M} \). The same idea can be applied to a situation where no data are available, but we are still interested in the uncertainty of a given focal mechanism. For example, we want to know how a given network resolves the moment tensor of offshore versus onshore earthquakes. Or we want to know how the resolvability of a deep event compares with that of a shallow event. Or we want to estimate the error of a solution reported by an agency, but we do not have the actual data. Or we plan a future deployment and want to know what station configuration is best. In these situations we need to evaluate the uncertainty (or resolvability) of a given focal mechanism without data. The family of acceptable solutions \( (\Delta \chi^2 < \Delta \chi^2_{threshold}) \) can be defined by prescribing \( \sigma_d^2 \) and \( \Delta \chi^2_{th} \), exactly as explained in **Best Solution versus Range of Acceptable Solutions**. However, if we have no data, how can we find a family of acceptable solutions? The theory of linear inverse problems has a simple answer for such a question: Errors of the model parameters are related to data errors through the Green’s functions matrix, \( \bf{G} \). Therefore, we can mathematically analyze a given source-station configuration, prescribe \( \sigma_d^2 \) and \( \Delta \chi^2 \), and compute the uncertainty of a given focal mechanism. The data are actually not needed to compute parameter uncertainties.

In moment tensor inversions, we need to determine six model parameters, \( m_1, \ldots, m_6 \) (equations 1, 2). Thus, the theoretical uncertainty region is six-dimensional \( (6D) \) (e.g., **Press et al., 1992**). In linear problems, the surfaces of constant L2 misfit (e.g., constant \( e^2 \), \( \chi^2 \), \( \Delta \chi^2 \), or \( VR \)) are ellipsoids in the 6D model space. The use of an L2 formulation implicitly assumes that the data errors are Gaussian (i.e., normally distributed). The shape of the 6D region will, in general, be more complex if data errors are non-Gaussian or if the problem is nonlinear. The reader is probably familiar with the error ellipsoids used for earthquake locations. A couple of differences between the error ellipsoids of earthquake locations and those of moment tensors are worth mentioning: The main difference is that the location of earthquakes is a nonlinear problem; thus, the location error ellipsoids are of limited interest (they are valid only for the linearized problem). The proper assessment of location errors requires the computation of nonelliptical 3D error volumes (e.g., **Lomax, 2005**). On the other hand, the error ellipsoids of \( \bf{M} \) cannot be fully visualized because they are six-dimensional. Although available in closed mathematical form, the error ellipsoid of
$\mathbf{M}$ has to be transformed into more meaningful quantities in order to become useful.

Throughout this paper we will present uncertainty regions, or error ellipsoids, rather than confidence regions as strictly defined in a statistical sense. The $\Delta \chi^2$ error ellipsoid can be translated into a confidence region (e.g., a 90% confidence region) only if data errors are normally distributed and their exact absolute value is known. In such cases, one can compute confidence regions given a choice of $\Delta \chi^2$. We refer the reader to section 15.6 of Press et al. (1992) for more details on how to compute ellipsoids of different confidence levels. Next, we briefly explain how to compute the error ellipsoid of $\mathbf{M}$: the Appendix contains a full-detail description of the algorithm.

The errors of the model parameters $\sigma(\mathbf{m})$ depend exclusively on the Green’s functions $\mathbf{G}$, on the data variance $\sigma_d^2$, and on the chosen $\Delta \chi^2$. In fact, the shape and orientation in model space of the error ellipsoid is computed from the eigenvalues and eigenvectors of the matrix $\mathbf{G}^T \mathbf{G}$ (Fig. 1; see the Appendix). Data variance ($\sigma_d^2$) and $\Delta \chi^2$ scale the size of the ellipsoid. For a fixed value of data error, the larger $\Delta \chi^2$ is, the larger the error ellipsoid and the more solutions are considered acceptable. Also, for a fixed limiting value of $\Delta \chi^2$, the larger the data variance $\sigma_d^2$, the larger the error ellipsoid. In order to compute the error of the model parameters $\sigma(\mathbf{m})$, we need the following information: location of the earthquake (epicenter and depth), station coordinates, crustal model and frequency band (all these parameters enter the computation of $\mathbf{G}$), an estimate of data error ($\sigma_d$), and a choice of $\Delta \chi^2$. We use the discrete wavenumber method, which takes as input 1D-layered models of velocity-density-attenuation, in order to compute Green’s functions (Bouchon, 1981; Coutant, 1989). Data error will be addressed in a subsection of its own (see Data Error). In this paper we will use $\Delta \chi^2 = 1$; this choice is arbitrary as we will not associate $\Delta \chi^2$ with a specific confidence level. However, we recall that setting $\Delta \chi^2 = 1$ is equivalent to accepting solutions whose squared misfit is larger than the minimum squared misfit by an amount equal to the data variance (eq. 7).

The error ellipsoid is centered at a point, $\mathbf{m}_{\text{ref}}$, of the parameter space. In the situation where we have data, $\mathbf{m}_{\text{ref}}$ corresponds to the optimal solution ($\Delta \chi^2 = 0$). If we do not have actual data, $\mathbf{m}_{\text{ref}}$ is the focal mechanism (or $\mathbf{M}$) whose resolvability we wish to assess. Let us consider the ellipsoid defined by $\Delta \chi^2 \leq 1$. The family of acceptable solutions inside the ellipsoid, including its surface, is defined as

$$\mathbf{m}_{\text{acceptable}} = \mathbf{m}_{\text{ref}} + \Delta \mathbf{m}. \quad (11)$$

In equation (11) $\mathbf{m}_{\text{acceptable}}$ is an acceptable solution, and $\Delta \mathbf{m}$ is a radius vector from the center of the ellipsoid to any point inside or on the surface of the ellipsoid. In order to numerically find the family of acceptable solutions, we simply perform a 6D-grid search around $\mathbf{m}_{\text{ref}}$, and find the solutions for which $\Delta \chi^2 \leq 1$. Thus, we determine which $\mathbf{b} \mathbf{m}_j$ (variations of $\mathbf{m}_j$ around $\mathbf{m}_{\text{ref}}$) fall inside the ellipsoid for all $j = 1, \ldots, 6$. In other words, we determine the amount by which we can vary the source parameters, $\mathbf{b} \mathbf{m}_j$, without leaving the $\Delta \chi^2 \leq 1$ region. The same 6D ellipsoid (set of points $\Delta \mathbf{m}$) is valid for investigating the uncertainty of any $\mathbf{m}_{\text{ref}}$, because the inverse problem is linear in $\mathbf{M}$. That is, we can use the same error ellipsoid to study the resolvability of an earthquake with any focal mechanism for a given hypocenter and station configuration (notice that the actual knowledge of $\mathbf{M}$ is not required to compute the error ellipsoid). This method of exploring the parameter space is considerably faster than other sensitivity studies which need to use seismograms and compute misfits.

The 6D error ellipsoid contains information about the uncertainty of the full moment tensor, including trade-offs between individual components of $\mathbf{M}$. We then have to translate these uncertainties into meaningful quantities. We will focus on the DC component of $\mathbf{M}$ as described by the strike, dip, and rake angles. The relation between $\mathbf{M}$ and strike, dip, and rake is nonlinear. Therefore, in order to compute a family of acceptable solutions in terms of strike, dip, and rake, we must choose a source mechanism $\mathbf{m}_{\text{ref}}$. The family of acceptable DC solutions can be displayed using nodal lines. In addition, we use Kagan’s angle (Kagan, 1991; hereafter, the $K$-angle), in order to characterize the family of acceptable solutions. The $K$-angle is the smallest rotation between two orthogonal bases describing different DCs (e.g., the smallest

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**Figure 1.** The shape and orientation of the error ellipsoid is determined by the singular vectors $\mathbf{v}_i$ and singular values $w_i$. The singular vectors $\mathbf{v}_i$ determine the direction of the ellipsoid axes; that is, they determine the orientation of the ellipsoid in the 6D-parameter space. The singular values determine the size of the ellipsoid axes, thus the shape of the ellipsoid. Data error, $\sigma_d$, scales the size of the whole ellipsoid. The surface of the ellipsoid is characterized by $\Delta \chi^2 = \text{constant}$. See the Appendix for more details. Figure based on Press et al. (1992).
angle between the $P$, $T$, and $B$ bases of two DCs; or equivalently, the smallest angle between the slip vectors of two DCs). In our application, the $K$-angle corresponds to the smallest amount by which we have to rotate each acceptable solution in order to get $m_{\text{ref}}$. Although less intuitive than strike, dip, and rake, the $K$-angle conveniently quantifies the deviation of the family of acceptable solutions from $m_{\text{ref}}$ using one single value. The $K$-angle is especially useful to plot maps of resolvability, which show the geographical variation of DC resolvability (in this case, we need one single parameter to characterize the resolvability on each grid point; see Network Design and Maps of Resolvability).

### Data Error

A key parameter in the estimation of the uncertainty of $M$ is data error. In principle, one can never know the exact value of data error. In this subsection we discuss the order of magnitude of $\sigma_d$. Let us start by considering the situation in which we do have a dataset to perform the inversion. Vasco (1990) proposed that the value of the data error be the maximum between the noise background level ($\sigma_n$) and the misfit between observed and synthetic waveforms:

$$
\sigma_d = \max(\sigma_n, |d - s|).
$$

(12)

The noise term, $\sigma_n$, generally produces values for data error on the order of $1\%$–$10\%$ of the data amplitude (noisier data are normally left out of the inversion). The term arising from the misfit between data and synthetics ($|d - s|$) can lead to considerably higher estimates of $\sigma_d$.

Note that data error and modeling error are two different error sources that should be treated independently (e.g., Tarantola, 1987; Press et al., 1992). Both types of errors determine the tolerable variation of model parameters simultaneously but in different ways. However, an independent assessment of data and modeling error is a difficult task. In order to have a true measure of data error, we would need to be able to repeat the measurements, that is, record the same earthquake, at the same sites, repeatedly. In order to have a true measure of model error, we would need to know the true uncertainty in crustal structure, etc. While the first term in equation (12) concerns data error strictly, the second term involves both data error and modeling error. The approach of Vasco (1990) recognizes that, in practice, error can only be quantified through the difference between data and synthetics. Monelli et al. (2009) also highlighted the difference between data error and modeling error, but given the impossibility to compute either, opted for an empirical approach where the error was obtained from the data-synthetics misfit. Given that (1) data and modeling errors cannot be accurately and independently assessed, and (2) the theory used in this paper only accounts for data errors, we will proceed by using a single formal quantity, hereafter called data error, to characterize the tolerated misfit between observed and synthetic waveforms (regardless of whether the data-synthetics mismatch arises from erroneous Green’s functions or seismograms).

Let us now consider the case in which we do not have data, but still want to assess the resolvability of $M$. In this case we can estimate the order of magnitude of expectable data error from empirical arguments. Let the data error, as defined by its standard deviation $\sigma_d$, be related to data amplitude by

$$
\sigma_d = C \cdot G \cdot M_0,
$$

(13)

where $C$ is a constant, $G$ is the peak displacement of the medium in response to a unit-moment source dislocation (in the studied source-station configuration and frequency band), and $M_0$ is the seismic moment of the earthquake whose resolvability we wish to assess. Note that $G \cdot M_0$ is an estimate of the data amplitude. Thus, if $C = 1$, then $\sigma_d$ is of the same order of magnitude as (synthetic) waveform data. What is a realistic estimate of $C$?

Let us analyze how popular choices of the variance-reduction threshold translate into $C$. Equation (9) shows that, for $\Delta \chi^2 = 1$, $\sigma_d^2$ is related to the data power $\sum d_i^2$ through the variance-reduction difference, $\text{VR}_{\text{min}} - \text{VR}_0$. Consider the following thought experiment: We inverted a dataset, found the strike, dip, and rake angles through grid search, and defined a family of acceptable solutions by prescribing a variance-reduction threshold, $\text{VR}_0$. Specifically, assume that $\text{VR}_{\text{min}} = 0.7$, and we chose $\text{VR}_0 = 0.6$. According to equation (9), accepting such values for the variance reduction is equivalent to estimating the magnitude of data variance:

$$
\sigma_d^2 = (0.7 - 0.6) \sum_i d_i^2.
$$

(14)

Recall that the summation of the data $d_i$ is made for all stations, components, and times. Assume that we have seven stations, and three waveforms were recorded at each station. Further assume that the dominant wave-group in the ground displacement is a 10-s triangle with peak value, $P$. Then,

$$
\sigma_d^2 = 0.1 \sum_i d_i^2 = 0.1 \times 3 \times 7 \times \frac{10}{2} P^2 = 10.5 \times P^2,
$$

(15)

and

$$
\sigma_d = 3.2 \times P.
$$

(16)

Thus, $\sigma_d$ is proportional to the data amplitude ($\sigma_d \propto P$), in good agreement with equation (13): $\sigma_d \propto G \cdot M_0$. In fact, equation (16) is equivalent to (13) if $C = 3.2$. Choosing a family of acceptable solutions via prescription of $\text{VR}_0$ close to $\text{VR}_{\text{min}}$ is equivalent to prescribing a value for data error of the same order of magnitude as the data itself. Note that in the example stated previously data error was even larger than the data amplitude ($C > 1$). The exact value of $C$ depends on the exact value of $\text{VR}_0$, the number of waveforms $N$, and on the source time function. This example shows that choosing $\sigma_d^2$ lower than the data amplitude is
probably too optimistic. Note in this example we did not assess the true value of data error. Rather, we showed that one popular choice of variance-reduction threshold (VR\textsubscript{min} = 0.7 and VR\textsubscript{th} = 0.6) is an implicit assumption that data error is of the same order of magnitude as the data itself (for \(\Delta x^2 = 1\)). The same is true for several other popular choices of the variance-reduction threshold, for example, (VR\textsubscript{min}, VR\textsubscript{th}) = (0.9, 0.8) or (0.4, 0.3).

A different approach: Consider identical waveforms shifted by \(\tau\) with respect to each other. For simplicity, assume that the identical waveforms are harmonic and their period is \(T\). The mismatch, \(\epsilon\), between the two waveforms is then given by

\[
\epsilon = d(t) - d(t - \tau) \approx \dot{d}(t)\tau = d(t) \frac{2\pi}{T} \tau.
\] (17)

If the difference between waveforms, \(\epsilon\), is taken as a representation of the data error \(\sigma_d\), then \(\sigma_d\) is proportional to the data itself, \(d(t)\). In this case, \(C = (2\pi/T)\tau\). Data error is smaller than the data amplitude (\(C < 1\)) only for small time shifts, \(\tau < T/2\pi\). Although extremely simple, this example is very relevant for inversions of \(M\). In fact, time shifts between the dominant quasi-periodic wave-group of data and synthetics are common in moment-tensor inversions. In opposition, the amplitude of the observed and synthetic waves is normally well matched. The time shifts between data and synthetics are commonly attributed to inaccurate crustal models or mislocations of the earthquake hypocenter. Alternatively, the time shifts may arise from no GPS synchronization of the station crystal clock. The former is a typical modeling error, and the latter is an example of true data error. Nevertheless, as explained previously, we do not distinguish the two types of error formally. Some practitioners introduce artificial time shifts to align the observed and synthetic seismograms prior to the inversion. If \(\tau > T/2\pi\), such alignment corresponds to declaring \(C > 1\), that is, that the data error is larger than the data itself (e.g., Ford et al., 2009) considered time shifts up to half-cycle \(T/2\).

Finally, what is the amplitude of the data error if the sensor is not properly oriented: a typical data error in its most strict sense? Ekström and Busby (2008) showed that the horizontal components of seismometers are typically rotated with respect to the true geographic coordinates by up to \(10^\circ\), some sensors presenting rotations up to \(30^\circ\). In this example, we consider waveforms of an \(M_w\) 6 earthquake recorded at four stations located approximately 200 km to 400 km away from the epicenter (see Reference Setup for more details). We rotated the horizontal components of ground motion up to \(30^\circ\) (Fig. 2a,c). We then subtracted the original trace from the rotated traces (Fig. 2b,d). Figure 3 shows the ratio between the maximum amplitude of data error, taken as the difference between rotated and original waveform, and the maximum amplitude of the data (original waveform). Data error varies between approximately 10% and 40% of the data amplitude for a \(10^\circ\)-sensor misorientation. For sensors misoriented by \(30^\circ\), data error varies between 30% and nearly 90% of the data amplitude.

The previous examples demonstrate that realistic estimates of the data error must be relatively large and comparable to data amplitude. Thus, an appropriate choice for the applications presented in this paper is \(C \geq 1\).

**Applications to Earthquakes in Southwest Iberia**

We now present examples of the assessment of DC resolvability for earthquakes on the active plate boundary between Africa (Nubia) and Eurasia, southwest of Portugal (e.g., Buforn et al., 1988; Serpelloni et al., 2004; Borges et al., 2001; Stich et al., 2003; Buforn et al., 2004; Stich et al., 2006, 2010) that commonly occur at depths down to 60 km (Grimison and Chen,
In the remainder of the text we will simply use the expressions resolvability or resolvability of DC to refer to the resolvability of the DC component of $M$. We investigate the effect of earthquake depth, focal mechanism, station configuration, crustal structure, and frequency band on the resolvability of a DC. All the results are summarized in Table 1. We also assess the sensitivity of the results to the choice of data error, $\sigma_d$, given a fixed value of $\Delta \chi^2 = 1$. Finally, we compute maps of resolvability for different station networks. As an example of an application using real data, we study the feasibility of single-station inversions, both using real waveforms and using our theoretical approach. All the studied cases were designed to mimic real situations.

Reference Setup

We will start by studying the resolvability of a DC source for a reference configuration as determined by earthquake location (epicenter and depth), focal mechanism, crustal structure, frequency band, station network, and data error (with fixed $\Delta \chi^2 = 1$). We will then change these parameters one by one to investigate their effect on DC resolvability. The reference configuration is defined as follows: The station network is composed of seven land stations located in Portugal, Spain, and Morocco (hereafter named the IB network). These seven stations are: PMAFR, PFVI, PNCL, and PBDV in Portugal; SFS in Spain; and RTC and NKM in Morocco. Many more broadband stations are currently installed in Iberia and northern Africa (Fig. 4). We chose to use a limited set of stations in order to show a simple application of the method. The analysis with a reduced set of stations is also instructive to learn about the resolvability of DCs in less favorable situations, in which only a few stations are available. The work is composed of seven land stations located in Portugal, Spain, and Morocco (hereafter named the IB network). These seven stations are: PMAFR, PFVI, PNCL, and PBDV in Portugal; SFS in Spain; and RTC and NKM in Morocco. Many more broadband stations are currently installed in Iberia and northern Africa (Fig. 4). We chose to use a limited set of stations in order to show a simple application of the method. The analysis with a reduced set of stations is also instructive to learn about the resolvability of DCs in less favorable situations, in which only a few stations are available. The

![Figure 3](image1.png)

**Figure 3.** Ratio between the maximum amplitude of data error (taken as the difference between rotated and nonrotated waveforms) and maximum amplitude of data (nonrotated waveform). Data error varies between approximately 10% and 40% of the data amplitude for a 10° misorientation of the seismometer. For sensors misoriented by 30°, the error varies between 30% and 90% of the data amplitude. The different lines represent different station-components. The color version of this figure is available only in the electronic edition.

![Figure 4](image2.png)

**Figure 4.** Map of the seismicity and seismic network in southwest Iberia. The star marks the reference earthquake epicenter. In the applications presented we study the resolvability of network PT (squares), network IB (squares and hexagons), network IB + OBS (squares, hexagons and circles) and network IBdense (diamonds). The figure also displays permanent stations (small solid triangles) and temporary stations, current or past (small transparent triangles). Small gray dots show the background seismicity in the instrumental record (post-1960) (Peña et al., 2011; Carrilho et al., 2004; Preliminary Seismic Information, 2010). Large dots show the locations of $M_w > 3.5$ earthquakes from 2007 to 2010. The four different focal mechanisms studied are plotted in the upper left corner, as well as their strike, dip, and rake values. Stations labeled in gray were not used in the estimate of data error due to sensor misorientation presented in Data Error, nor in the single-station inversions presented in Application with Real Data: Single-Station Inversions, because data was either not available or too noisy/inconsistent. The color version of this figure is available only in the electronic edition.
reference earthquake is located offshore, on the Horseshoe Abyssal Plain (HAP), at 35.90° N and 10.31° W (Fig. 4). This location was chosen after the epicenter of an $M_w$ 6 earthquake that occurred on 12 February 2007 (Stich et al., 2007; Custódio et al., 2012). The reference epicenter is located in a region of active seismicity (e.g., Borges et al., 2001; Stich et al., 2003; Buform et al., 2004; Serpelloni et al., 2007; Geissler et al., 2010 and references therein). The reference source depth is 40 km, also chosen after the epicenter of an $M_w$ 6 earthquake that occurred on 12 February 2007 earthquake. Earthquakes occur frequently at reference source depth is 40 km, also chosen after the 12 February 2007 earthquake. Earthquakes occur frequently at these depths of 40 km below the HAP (Grimison and Chen, 1986; Stich et al., 2007, 2010; Geissler et al., 2010). The crustal structure is described by the 1D layered model proposed by Stich et al. (2003) for the Hercynian basement, Mesozoic platforms and mixed propagation paths. The reference frequency band is 0.025 Hz to 0.08 Hz. The reference focal mechanism, $m_{ref}$, is described by scalar moment, $M_0 = 1 \times 10^{16}$ N · m, and strike, dip, and rake angles equal to 128°, 46°, and 138°, respectively (or equivalently 250°, 61°, and 52°). We assume that data error, $\sigma_d$, is equal to $10^{-5}$ m. This value is obtained using equation (13): We take the seismic moment $M_0 = 1 \times 10^{16}$ N · m, compute $G (\sim 10^{-21}$ m/N · m), and set $C = 1$. Note that all the reference earthquake parameters are chosen as those of the $M_w$ 6 earthquake that occurred on 12 February 2007, offshore southwest Iberia (Stich et al., 2007; Buform et al., 2007; Carrilho et al., 2007; Custódio et al., 2009). The one exception is the seismic moment, which was set to $M_0 = 1 \times 10^{16}$ N · m, corresponding to an $M_w$ 4.6 event. This magnitude was chosen as the average magnitude of events that occur offshore southwest Iberia and which are studied in MT inversions.

We start by determining the $\Delta \chi^2 < 1$ error ellipsoid for the reference configuration according to the algorithm outlined in the Appendix. Once we have obtained the error ellipsoid, we find the corresponding acceptable values of the faulting angles of strike, dip, and rake: (1) we take all the (discrete) points inside the ellipsoid, $m_{acceptable}$, in equation (11); (2) we convert $m_{acceptable}$ into actual moment tensors, $M_{acceptable}$, using equation (2); and (3) we convert $M_{acceptable}$ into acceptable values of strike, dip, and rake using the standard relations between $M$ and faulting angles. This simple transformation allows an intuitive visualization of the family of acceptable solutions. Figure 5a–c shows the faulting angles plotted as a function of the misfit, $\Delta \chi^2$. The left and right edges of the plots correspond to the center ($\Delta \chi^2 = 0$) and surface ($\Delta \chi^2 = 1$) of the ellipsoid, respectively. The two strips in each panel correspond to the two DC conjugate solutions (two possible faulting planes). Note that we find solutions close to the reference values of strike, dip, and rake, even on the surface of the ellipsoid (we will see further ahead that they do not occur simultaneously though). Figure 5d shows the same information as panels 5a–c displayed in a different format: The deviations of strike, dip, and rake from the reference solution are now quantified with the $K$-angle. The $K$-angle describes in one single value the deviation of a given solution from the reference solution. The histograms of Figure 5e–h show the statistical distribution of strike, dip, and, the $K$-angle. Note that the histograms of the strike, dip, and rake angles are peaked at their reference values (one peak for each conjugate solution). However, the $K$-angle is not peaked at zero, as would be expected if the most frequently occurring solution were $m_{ref}$. The Appendix explains that most points detected by our grid search inside the error ellipsoid are located on, or very close to, the ellipsoid surface. This sampling of the error ellipsoid results from the uniform sampling of the
6D-parameter space. The reference values occur individually but not simultaneously on the surface of the ellipsoid. In fact, the reference values occur simultaneously only at the center of the ellipsoid.

For the reference configuration, the mean value of the $K$-angle is $K_{\text{mean}} = 14^\circ$, its maximum value is $K_{\text{max}} = 34^\circ$, and its standard deviation is $\sigma(K) = 6^\circ$. These values are not meaningful in an absolute sense, because they depend on our

Figure 5. (a–c) Strike, dip, and rake angles for the family of acceptable solutions versus misfit, $\Delta x^2$, for the reference setup: The source-station geometry of network IB is shown in Figure 4; source depth is 40 km; strike, dip, rake angles are 128°, 46°, 138°, equivalent to 250°, 61°, 52°. (d) $K$-angle corresponding to the focal mechanisms shown in panels (a–c). (e–h) Histograms of strike, dip, rake, and the $K$-angle. The color version of this figure is available only in the electronic edition.
order-of-magnitude estimate of data error rather than on an absolute value of data error. However, they are useful in a relative sense, as they allow us to compare the resolvability of a given focal mechanism in different configurations of the inverse problem. We will mostly use $K_{\text{mean}}$ throughout the paper in order to compare the DC resolvability in different situations. $K_{\text{mean}}$ measures the deviation of the acceptable solutions (mostly located on the surface of the $\Delta \chi^2 = 1$ ellipsoid) from $m_{\text{ref}}$.

Source Depth

We will now repeat the previous analysis using two different hypocenter depths: a shallow 10-km-source and a moderate-depth 60-km-source. Figure 6 shows that the uncertainty of the focal mechanism increases dramatically with source depth. The mathematical explanation for such decrease is that different source-station configurations have different Green’s functions (matrix $G$); hence, they also have different singular vectors and singular values, and different error ellipsoids. The physical explanation is not as straightforward; nevertheless, it is understandable that deeper sources produce a simpler wave field, with weaker $L_g$ waves. Thus, seismograms from deep sources carry less, or lower-amplitude, information about the source. Note that the excellent resolvability of the shallow source (10 km) assumes that we know the exact Green’s functions. The uncertainty of the DC solution for a 10-km-deep source is underestimated by our analysis if the shallow crustal structure is less well known than its deeper counterpart.

Station Network

We now test the resolvability of DC parameters using different station networks. The earthquakes that happen offshore southwest Iberia are normally recorded by stations in Iberia and northern Africa; however, a large azimuthal gap does exist, which may hinder the studies of seismic sources. In order to improve our knowledge of the local earthquake activity one might think of deploying a permanent OBS network. It is interesting to investigate how the resolvability of $M$ would improve using such an OBS network. Thus, we tested the following station configurations (Fig. 4): (1) Network IB (the already-described reference configuration) with seven land stations distributed in Portugal, Spain, and Morocco; (2) Network PT, composed of only four stations in Portugal; (3) Network IB + OBS, composed of the IB land stations plus four OBS stations; and (4) Network IBdense, composed of the IB network plus 23 land stations along the coastlines of Portugal, Spain, and Morocco, totalling 30 stations. The location of the four OBS stations were chosen as the true locations of temporary OBS stations (Geissler et al., 2010). Figure 7 shows that, for the reference network IB, the family of acceptable solutions has a mean deviation from the true focal mechanism, $K_{\text{mean}} = 14^\circ$. $K_{\text{mean}}$ increases to $22^\circ$ when we use the reduced land network PT, corresponding to a strong decrease in the resolvability of DCs. The resolvability in turn dramatically improved when we add four OBS stations to the IB network, with $K_{\text{mean}}$ dropping to $6^\circ$. A similar improvement in the resolvability is achieved with a densification of the land network ($K_{\text{mean}} = 8^\circ$ for network IBdense). Notice that the resolvability of the focal mechanism improves using IBdense, in spite of the large azimuthal gap. The resolvability of a 10-km-deep event using the IB network is similar to that of a 40-km event using IB + OBS or IBdense; in both cases the resolvability of the DC is good. On the other hand, the resolvability is poor for a 60-km-deep event using network PT, equalling the resolvability of a 40-km-deep event using network PT.

![Figure 6](image_url)

Figure 6. (a–c) Histograms of the $K$-angle indicating the resolvability of a DC source for three different source depths: 10, 40, and 60 km. The mean (and maximum) values of the $K$-angle are $6^\circ$ ($14^\circ$), $14^\circ$ ($34^\circ$), and $22^\circ$ ($91^\circ$), respectively. Dashed lines indicate the average $K$-angle. (d–f) The same results plotted using the more familiar nodal lines. Note the excellent resolvability of the shallow source (10 km). The color version of this figure is available only in the electronic edition.
Frequency Range

So far we worked with waveforms in the 0.025–0.08 Hz frequency range. If we increase the low-frequency limit from 0.025 Hz to 0.05 Hz (i.e., if we remove the low frequencies from the inversion), the resolvability deteriorates, with $K_{\text{mean}}$ rising to 25°. This is exactly what happens in real inversions of weak events, where the low frequencies cannot be used due to their low S/N ratio. On the contrary, if we use higher frequencies (increase the upper frequency limit from 0.08 Hz to 0.2 Hz), the resolvability improves slightly, with $K_{\text{mean}}$ dropping to 13° and $K_{\text{max}}$ dropping to 26°. However, the resolvability improvement based on high frequencies is of little usage. In practice, the existing crustal models do not always provide realistic Green’s functions at near-regional distances up to a frequency of 0.2 Hz.

Crustal Structure

We can formally test different crustal models while keeping the source-station configuration and frequency range unchanged. In this situation we alter the matrix $G$, thus changing the error ellipsoid. We tested four different crustal structures: (1) the regional model of Stich et al. (2003), used in the reference setup; (2) the global preliminary reference earth model (PREM) (Dziewonski and Anderson, 1981); (3) a homogeneous, fast, low-attenuation, mantle-like, homogeneous half-space, with $V_p = 8.1$ km/s, $V_S = 4.5$ km/s, $Q_P = 1340$, and $Q_S = 600$; and (4) a homogeneous, slow, high-attenuation, crust-like, homogeneous half-space, with $V_p = 5.8$ km/s, $V_S = 3.2$ km/s, $Q_P = 300$, and $Q_S = 150$. $K_{\text{mean}}$ for each of the four models is 14°, 14°, 31°, and 16°, respectively. It is interesting to note that the resolvability of the DC source mechanism is very similar, whether the true crustal structure resembles PREM, the regional model of Stich et al. (2003), or a crust-like homogeneous half-space. However, the resolvability decreases strongly if the medium is fast, in spite of low-attenuation (case 3).

The results stated previously concern the resolvability of a DC mechanism if the true crust is similar to that of each studied model. These results must not be confused with the investigation of the effect of an erroneous crustal structure upon the uncertainty of the source parameters.

Focal Mechanism

A given source-station configuration provides different $K$-angle distributions for different reference focal mechanisms, due to the previously mentioned nonlinearity of the inverse problem with respect to faulting angles. We tested the DC resolvability of four typical focal mechanisms in the study region (Fig. 4): (1) strike, dip, and rake angles equal to 128°, 46°, 138° (reference setup); (2) 52°, 57°, 99°; (3) 204°,

Figure 7. Resolvability of the 40-km-deep DC source using four different station networks (Fig. 4): (a) PT network (squares), (b) IB network (squares and hexagons), (c) IB + OBS network (squares, hexagons, and circles), (d) IBdense network (diamonds). The resolvability of this offshore DC source is very similar whether we use IB + OBS or IBdense. The color version of this figure is available only in the electronic edition.
68°, −16°; (4) 316°, 35°, −170°. The mean K-angles obtained for each case are 14°, 14°, 16°, and 16°, respectively. These results indicate that $K_{\text{mean}}$, and therefore the DC resolvability, do not vary significantly with focal mechanism.

Data Variance

All the previous calculations assumed a constant value for the data variance, $\sigma_d^2$. In particular, we assumed $\sigma_d = 10^{-5}$ m, which we obtained by setting $C = 1$ in equation (13). When we evaluate (14), using synthetic data in order to compute $d_i$, we obtain $\sigma_d = 1.4 \times 10^{-5}$; that is, a similar value of $C = 1.4$.

Data variance plays a key role in our analysis, controlling the size of the error ellipsoid (see the Appendix). By prescribing a constant value for $\sigma_d^2$, we were able to compare the uncertainty of DCs for different configurations of the inverse problem. Figure 8 shows the sensitivity of the K-angle to data variance $\sigma_d^2$ in the reference setup. As expected, an increase in $\sigma_d^2$ leads to an increase in the K-angle. The K-angle is very sensitive to $\sigma_d^2$ in the range $10^{-10}$ to $10^{-8}$ m².

Network Design and Maps of Resolvability

With the tools presented in this paper, it is possible to compute maps of DC resolvability. In order to plot maps of resolvability, we perform the analysis presented previously for different source epicenters. We start by choosing a station network (as defined by station locations), crustal model, frequency range, earthquake depth, focal mechanism, and data variance. We then compute the uncertainty of strike, dip, rake, and finally the K-angle for several epicenters on a regular grid of source latitudes and longitudes. The maps of resolvability can be easily computed and allow an immediate visualization of the geographical variation of DC resolvability; these maps are a useful tool to plan seismic networks or experiments. Given a seismic network, the maps of resolvability are useful to know where the better-resolved DC solutions are located and where we have no resolution on DCs.

Figure 9 shows the geographical variation of the K-angle, indicating the resolvability of the DCs, for three different station configurations: IB, IB + OBS, and IBdense. The parameters used to compute the maps are those of the reference setup. The IB network is able to resolve the DC parameters of onshore earthquakes better than those of offshore earthquakes. The addition of OBS stations (IB + OBS) improves the resolvability of offshore sources. The dense land network, IBdense, is able to resolve DC parameters of offshore earthquakes as well as the network IB + OBS. IBdense further improves the resolvability of onshore earthquakes in comparison with IB and IB + OBS.

Application with Real Data: Single-Station Inversions

The method presented in this paper mainly serves the purposes of network design and comparative analysis of MT resolvability. However, the method may also be used to help understand the uncertainty of a particular inversion. In this final application we use data of the $M_w$ 6.0 earthquake of 12 February 2007 (see Reference Setup). Moment tensor inversions are performed using the software ISOLA (Sokos and Zahradník, 2008). We invert data recorded at four stations: PFVI, PBDV, PMAFR, and RTC (Fig. 4). These are the stations of network IB that recorded the earthquake and whose waveforms had good quality. The component PBDV-NS was removed from the inversion due to a long-period disturbance (Zahradník and Plešinger, 2005). The crustal model and epicenter used in the inversion are the same as that of the reference setup. Centroid depth is grid-searched in the range 25 km to 55 km in steps of 5 km. The origin (centroid) time is also grid-searched, which only shifts all waveforms by the same amount. The inversions are performed in two frequency ranges: 0.01–0.10 Hz and 0.05–0.10 Hz. Figure 10a,c shows the reference focal mechanism (thick full line) and the focal mechanism inferred from the inversion of the waveforms recorded at the four stations (thick dashed line) in the two frequency bands. In terms of the K-angle, the inferred focal mechanisms differ little (less than 10°) from the reference solution shown in Table 2.

We test the resolvability of single-station MT inversions in order to investigate whether the results of our theoretical resolvability analysis hold. Successful single-station inversions have been reported, for example, by Fan and Wallace (1991), Dreger and Helmberger (1993), and Kim and Kraeva (1999). However, single-station inversions remain rare and are often perceived with skepticism. Single-station inversions do deserve more attention because: (1) they oppose the myth that the study of focal-mechanisms requires a good azimuthal coverage, and (2) single-station inversions are important in forensic seismology, in future planetary missions, etc.
Let us first present the results of single-station inversions using real data, and then make the resolvability theoretical. We invert the waveforms of each single station individually, thus obtaining four different MT solutions. Their DC-parts can be compared with the reference solution in terms of the $K$-angle. We repeat this procedure in the two frequency ranges defined previously. The results are shown in Figure 10 and Table 2. In the presence of low frequencies (0.01–0.10 Hz), the inferred DCs form a relatively compact family of nodal lines close to the reference solution. The $K$-angles between the reference and single-station focal mechanisms fall in the range $[11°, 39°]$. The single-station inversions

Figure 9. Maps of $K$-angle indicating the resolvability of a DC source using different station networks: (a) IB; (b) IB + OBS; (c) IBdense. Dark cells indicate better-resolved DC sources. The stations that compose each network are shown by white triangles. This map concerns earthquake sources at depths of 40 km. Using OBSs in addition to the land IB network improves the resolvability of the DC source considerably, particularly for offshore sources. The dense land network, IBdense, provides a similar improvement of the DC resolvability of offshore earthquakes and further improves the resolvability of onshore earthquakes. The color version of this figure is available only in the electronic edition.

Let us first present the results of single-station inversions using real data, and then make the resolvability theoretical. We invert the waveforms of each single station individually, thus obtaining four different MT solutions. Their DC-parts can be compared with the reference solution in terms of the $K$-angle. We repeat this procedure in the two frequency ranges defined previously. The results are shown in Figure 10 and Table 2. In the presence of low frequencies (0.01–0.10 Hz), the inferred DCs form a relatively compact family of nodal lines close to the reference solution. The $K$-angles between the reference and single-station focal mechanisms fall in the range $[11°, 39°]$. The single-station inversions

Figure 10. Focal mechanisms and waveform fits obtained from the inversion of real data (waveforms) of the $M_w$ 6 earthquake of 12 February 2007. (a) Thick full lines display the reference focal mechanism, thick dashed nodal lines show the focal mechanisms inferred from the four-station inversion, and thin nodal lines were derived from single-station inversions. The $P$ and $T$ axes are shown by squares and circles, respectively. (b) Data (light) and synthetic (dark) waveforms generated by single-station inversions. The dashed waveform (PBDV NS) is excluded from the inversion due to a long-period disturbance. Panels (a) and (b) display the results of the inversions in the $[0.01, 0.1]$ Hz frequency range. Panels (c) and (d) are similar to panels (a) and (b), respectively, only for inversions in the $[0.05, 0.1]$ Hz frequency range. Note the considerably less scatter in single-station inversions when frequencies below 0.05 Hz were used (a). The color version of this figure is available only in the electronic edition.
are stable and provide acceptable results. In the frequency range 0.05–0.10 Hz, the nodal lines are considerably more scattered. The $K$-angles now fall in the interval [27°, 81°]. The single-station inversions are almost valueless. It is interesting to note that station PBDV provides a reasonable solution in both frequency bands even though we inverted only two components (EW and Z).

Finally, we perform the theoretical resolvability analysis without waveforms. The data standard deviation, $\sigma_d$, is estimated according to equation (13). We find $G \sim 3 \times 10^{-22} \text{ mN} \cdot \text{m}$ and $\sim 1.5 \times 10^{-22} \text{ mN} \cdot \text{m}$ for [0.01, 0.10] Hz and [0.05, 0.10] Hz, respectively. Plugging in $C = 1$ and $M_0 = 1 \times 10^{18} \text{ N} \cdot \text{m}$, we obtain $\sigma_d^2 = 9 \times 10^{-8} \text{ m}^2$ and $2.25 \times 10^{-8} \text{ m}^2$, respectively. The mean $K$-angles computed theoretically with this estimate of data error fall in the intervals [18°, 42°] for [0.01, 0.10] Hz and [31°, 54°] for [0.05, 0.10] Hz. The theoretical (eigenvector) analysis agrees well with the real data inversion and is consistent with the conclusions of Frequency Range: The use of a broad range of frequencies improves the resolvability. Our analysis corroborates the previous conclusion that single-station inversions should be successful when low frequencies are available, given the studied source-station configuration. However, single-station inversions are expected to fail when low frequencies are missing. This example illustrates how the theoretical resolvability analysis may help understand real data inversions by indicating under which circumstances the MT solutions are reliable.

### Discussion and Conclusions

In this paper we presented a method, based on the standard theory of inverse linear problems, to compute the uncertainty of the focal mechanisms inferred from the inversion of $\mathbf{M}$. The input parameters are the source location (epicenter and depth), stations locations, crustal model, frequency band of interest, and estimate of data error. The output is a 6D error ellipsoid, which contains a family of acceptable solutions in the space of model parameters, $m_{ij}$. This 6D ellipsoid shows the deviation of the acceptable focal mechanisms from the optimal (or reference) solution. The larger the data variance, $\sigma_d^2$, the larger the variability of the acceptable focal mechanisms. We restricted our analysis to the uncertainty of the double-couple component of $\mathbf{M}$, although the 6D error ellipsoid contains information about the uncertainty of all the components of $\mathbf{M}$. We mapped the uncertainties of the model parameters $m_{ij}$ into uncertainties of the faulting angles strike, dip, and rake. We used the $K$-angle to quantify with a single parameter the deviation of the acceptable DC solutions from the reference solution.

The resolvability of DC sources is determined by data error and by the eigenvectors and eigenvalues of the underlying inverse problem. These eigenvectors and eigenvalues are computed from the Green’s functions matrix, $\mathbf{G}$, and are thus tightly related to the source-station configuration. Some sets of eigenvectors and eigenvalues allow an adequate retrieval of the moment tensor $\mathbf{M}$, while others do not. Large eigenvalues correspond to robust source parameters. Small eigenvalues are associated with poorly resolved parameters. The problem is very similar to that of linear finite-fault inversions, although it uses a significantly smaller number of model parameters (Page et al., 2009; Custódio et al., 2012; Zahradník and Gallovič, 2010; Gallovič and Zahradník, 2011).

The method presented in this paper is applicable to linear problems only. The inversion of $\mathbf{M}$ is a linear problem assuming that the source location, origin time, and source time function are known. The algorithm can be generalized to the nonlinear case, where we search for the source position and origin time, in addition to $\mathbf{M}$. This generalization is the focus of ongoing work and will be presented in a separate paper. A nonlinear approach will allow the study of the trade-off between depth and non-DC components of $\mathbf{M}$. The uncertainty of the non-DC components can then be correctly estimated.

A proper assessment of data error $\sigma_d$ is required in order to determine the uncertainty of DC sources. We showed that realistic data errors should be of the same order of magnitude of the data itself, mostly due to inaccurate crustal models (and respective Green’s functions), but also due to instrumental errors.

Our method allows an estimation of the uncertainty of DC parameters if data is available. In this case we process the available data, solve the inverse problem, and obtain a minimum-misfit best solution. Then we prescribe a value for data error and estimate the uncertainty of each faulting angle. The method can also be applied when no data (waveforms) are available, which is particularly useful for network design. In this paper we present several examples for which an exact knowledge of data errors is not needed. We simply choose a reasonable value for $\sigma_d^2$, keep it fixed, and study the comparative resolvability of DC sources in different scenarios. In other words, we assess the resolvability of the focal mechanism in a relative sense. In particular, we assessed the relative effect of source epicenter, source depth, station location, crustal structure, frequency range, and data error on the resolvability of DCs. We focused on the resolvability of earthquakes in southwest Europe. A few stations were selected to represent the seismic networks of Portugal, Spain, and Morocco. We also considered a few OBS sites in the Atlantic Ocean. The analysis was performed in the frequency range below 0.1 Hz, which is typical for the near-regional inversions of $\mathbf{M}$. Table 1 summarizes the results.

We find that, for the same network, the resolvability depends critically on source depth. Shallow DC sources

---

### Table 2

<table>
<thead>
<tr>
<th>Frequency Range (Hz)</th>
<th>Four Stations Real Data</th>
<th>Single-Station Real Data</th>
<th>Single-Station Theoretical Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01–0.10</td>
<td>8°</td>
<td>[11°, 39°]</td>
<td>[18°, 42°]</td>
</tr>
<tr>
<td>0.05–0.10</td>
<td>6°</td>
<td>[27°, 81°]</td>
<td>[31°, 54°]</td>
</tr>
</tbody>
</table>
(10 km) are better resolved than deeper sources (40 and 60 km) by a factor of 2 or more. Our analysis indicates that the focal mechanism of a shallow offshore earthquake can be well resolved even by a small subset of land stations in Portugal despite the large azimuthal gap. These results are supported by unpublished inversions of different-depth earthquakes (Křížová-Červinková, 2008). The author repeated single-station waveform inversions using data from a near-regional network in Greece for the shallow Trichonis Lake earthquake (Kiratzi et al., 2008) and for the intermediate-depth Leonidio earthquake (Zahradník, Gallovič, et al., 2008). The focal mechanisms inferred from single-station inversions of the shallow event were all nearly identical. However, all single-station inversions of the intermediate-depth event failed.

Our uncertainty analysis also indicates that, if more stations are available, the resolvability of the 40- to 60-km-deep events can be as good as that of the shallow event using less stations. The resolvability of DCs improves considerably when the Portuguese network (PT) is supplemented by stations in Spain and Morocco (IB). Further improvement can be achieved by adding a few OBS stations offshore. Indeed, the theoretical DC resolvability of a 40-km-deep event using IB + OBS is almost as good as the resolvability of the 10-km-deep event using only the four stations of network IB. The improvement of the DC resolvability due to additional OBS stations is partly due to the decrease of the azimuthal gap. Yet we showed that a similar improvement in resolvability can be achieved with a densification of land stations (IBdense), where the azimuthal gap remains as large as ~200°. The resolvability of DC sources does not depend only on azimuthal coverage. A good azimuthal coverage is useful, but not always strictly necessary.

Finally, a word on the use of OBS waveforms in regional earthquake inversions is needed. A temporary deployment of broadband (BB) OBS (BB-OBS) stations recorded earthquakes offshore southwest Iberia within the scope of project NEAREST (Fig. 4). Geissler et al. (2010) used the first-motion polarities of the BB-OBS data in order to determine focal mechanisms of weak offshore events. The use of BB-OBS waveforms of regional earthquakes is challenging due to noise and various disturbances in the data, for example, tilts (Crawford and Webb, 2000). We attempted to use the NEAREST BB-OBS waveforms in a moment-tensor inversion: We started by taking the waveforms from a single $M_w$ 4.5 event, the largest event in the dataset, recorded at five three-component OBS stations. We then tried to remove the tilt-like disturbances from the horizontal components according to Zahradník and Plešinger (2005, 2010). Almost no signal was left in the frequency band of interest for near-regional waveform inversion after the successful removal of the tilt-like disturbances. The use of BB-OBS waveforms of regional earthquakes might be more successful for smaller source-station distances (< 100 km) and/or larger magnitude earthquakes.

As an example of an application using real data, we studied the $M_w$ 6 earthquake of 12 February 2007 located offshore Iberia. We performed four-station inversions in two different frequency ranges, which provided results similar to the previously published solutions. Then we repeated the inversions using data from single stations separately. The effect of frequency range became quite evident: Single-station inversions were successful only when lower frequencies were included. Or, more strictly, the single-station inversions that used a broader frequency range (0.01–0.10 Hz) resulted in less scattered faulting planes than those that lacked low frequencies (0.05–0.10 Hz). This behavior was confirmed by the theoretical calculation of the $K$-angle for single-station inversions in the two frequency ranges, without using any waveforms. These results attest to the applicability of the theoretical analysis outlined in this paper. The results also demonstrate that in waveform MT inversion, a good azimuthal coverage is not strictly necessary to obtain the correct strike, dip, and rake angles. More important than azimuthal coverage is a good S/N ratio in a sufficiently broad frequency band. On the other hand, a good azimuthal coverage is important for resolving the centroid position and/or the non-DC part of the MT. It should be noted that the results reported in this paper for single-station inversions depend on the studied source-station configuration and crustal model.

The implementation of the assessment of DC resolvability in the user-friendly software ISOLA is currently underway. The ISOLA package combines Fortran codes that perform moment-tensor inversion with a MATLAB GUI interface (Data and Resources).

**Data and Resources**

Seismic waveforms were kindly made available by Instituto de Meteorologia, Lisbon (Portugal), Real Instituto y Observatorio de la Armada, San Fernando (Spain) and Centre National pour la Recherche Scientifique et Technique, Rabat (Morocco). The software ISOLA (Sokos and Zahradník, 2008) was used to prepare the matrices for the computations of resolvability (http://seismo.geology.upatras.gr/isola/, last accessed January 2012). Green’s functions in ISOLA were computed using the AXITRA code of Coutant (1989). The Global Mapping Tools (GMT; Wessel and Smith, 1998) and MATLAB were used for figure plotting.

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**References**

Appendix

This appendix focuses on the theory and algorithm employed to construct the 6D error ellipsoid of \( \mathbf{M} \). We underscore that this analysis is standard and general for linear inverse problems. Further details on the method can be found, for example, in Menke (1989), Press et al. (1992), and Parker (1994).

Recall equation (1), which states the linear relation between ground motion \( \mathbf{d} \) and moment tensor parameters \( \mathbf{m} \), through the Green’s functions \( \mathbf{G} \):

\[
\mathbf{d} = \mathbf{Gm}. \tag{A1}
\]

In order to take into account data error, we divide both sides of (A1) by data standard deviation \( \sigma_d \):

\[
\tilde{\mathbf{d}}_i = \frac{d_i}{\sigma_d}, \tag{A2}
\]

\[
\tilde{\mathbf{g}}_{ij} = \frac{G_{ij}}{\sigma_d}. \tag{A3}
\]

Equation (A1) can then be rewritten as

\[
\tilde{\mathbf{d}} = \tilde{\mathbf{Gm}}. \tag{A4}
\]

where \( \tilde{\mathbf{G}} \) is the so-called design matrix. Solving (A4) in order to find \( \mathbf{m} \) is equivalent to minimizing \( \chi^2 \) (equation 5). The general least-squares (LSQ) solution of (A4) can be obtained with the aid of normal equations. In this case, the solution is given by

\[
\mathbf{m}_{\text{min}} = (\tilde{\mathbf{G}}^T\tilde{\mathbf{G}})^{-1}\tilde{\mathbf{G}}^T\tilde{\mathbf{d}}. \tag{A5}
\]

Alternatively, the solution can be obtained by means of singular value decomposition (SVD) of the matrix \( \tilde{\mathbf{G}} \):

\[
\tilde{\mathbf{G}} = \mathbf{U} \cdot \mathbf{W} \cdot \mathbf{V}^T. \tag{A6}
\]

Matrices \( \mathbf{U} \) and \( \mathbf{V} \) are orthonormal, and matrix \( \mathbf{W} \) is diagonal. The columns of \( \mathbf{U} \) and \( \mathbf{V} \) are the left- and right-singular vectors of \( \tilde{\mathbf{G}} \), respectively. Hereafter we use singular vectors to designate the columns of matrix \( \mathbf{V} \), which we represent by \( \mathbf{V}_i \). The diagonal elements of matrix \( \mathbf{W} \) are the singular values \( \omega_i \) of \( \tilde{\mathbf{G}} \). The model parameters \( m_j \) obtained through SVD are then

\[
\mathbf{m}_{\text{min}} = \sum_{i=1}^{6} \left( \mathbf{U}_i \cdot \frac{\tilde{\mathbf{d}}}{\omega_i} \right) \mathbf{V}_i. \tag{A7}
\]

where the sum runs over \( i = 1, \ldots, 6 \) model parameters. The parameter variances and covariances can be expressed in a very transparent form, using the singular vectors \( \mathbf{V}_i \) and singular values \( \omega_i \):

\[
\sigma^2(m_j) = \sum_{i=1}^{6} \left( \frac{\omega_j}{\omega_i} \right)^2 \tag{A8a}
\]

\[
\text{Cov}(m_j, m_k) = \sum_{i=1}^{6} \left( \frac{\omega_j \omega_k}{\omega_i^2} \right). \tag{A8b}
\]

Within the scope of this paper we will not use the parameter covariances. The covariance matrix contains explicit
information about the trade-off between every pair of model parameters. The information about parameter trade-off is also contained in the error ellipsoid. The error ellipsoid of $\mathbf{M}$ can be obtained from the singular vectors ($\mathbf{V}_{(i)}$) and singular values ($w_i$) of $\mathbf{G}$ (Fig. 1). The orthonormal singular vectors $\mathbf{V}_{(i)}$ define the direction of the ellipsoid principal axes. The length of the ellipsoid axes are inversely proportional to the singular values $w_i$. Thus, small singular values correspond to long axes of the ellipsoid (poorly resolved parameters). The most robust parameters are those associated with large singular values $w_i$. The actual size of the ellipsoid is determined by $\mathbf{V}_{(i)}$ and $w_i$. The shape and orientation of the error ellipsoid is fully determined by $\mathbf{V}_{(i)}$ and $w_i$. The surface of the ellipsoid, defined as a surface of constant $\Delta \chi^2$, can then be written as (Press et al., 1992)

$$\Delta \chi^2 = w_1^2(V_{(1)} \cdot \delta \mathbf{m})^2 + \ldots + w_6^2(V_{(6)} \cdot \delta \mathbf{m})^2. \quad (A9)$$

In (A9) $\delta \mathbf{m}$ is the vector between the center of the ellipsoid and a point in the parameter space.

Once we know the singular vectors $\mathbf{V}_{(i)}$ and singular values $w_i$, we can easily determine numerically the error ellipsoid: We set the size of the ellipsoid by choosing the limiting $\Delta \chi^2$ value, hereafter $\Delta \chi^2 = 1$. Then we find the model parameters $\delta m_j$, which are within the ellipsoid $0 \leq \Delta \chi^2 \leq 1$. In practice, we discretize $\delta m_1$, $\delta m_2$, $\ldots$, $\delta m_6$, and grid-search the 6D-parameter space within the limits given by $\pm \sigma(m_1), \ldots, \pm \sigma(m_6)$, calculated from (A8). Note that the surface $\Delta \chi^2 = 1$ is chosen arbitrarily, not associated with a specific confidence level. The grid search is fast, requiring only a few seconds on a laptop (it can be relatively coarse, for example, 11 points for each axis, 6 nested loops, totalling $11^6$ numerical evaluations of A9).

Figure A1 shows selected 2D cross-sections of $\Delta \chi^2 = 1$ error ellipsoids along axes $m_1$ and $m_2$. The center of the ellipsoid ($\Delta \chi^2 = 0$) is the reference solution $\mathbf{m}_{\text{ref}}$. The discrete points inside the ellipsoid are the family of acceptable solutions, $\mathbf{m}_{\text{acceptable}} = \mathbf{m}_{\text{ref}} + \delta \mathbf{m}$. Note that the projection of the ellipsoid onto the individual axes equal the individual parameter standard deviations, $\sigma(m_j)$. This is a consequence of the adopted choice of limiting $\Delta \chi^2 = 1$. The ellipsoid cross-sections allow a visualization of the well-known trade-offs between model parameters. However, the practical interest of this visualization is limited, because: (1) we have to consider all six dimensions and (2) the model parameters do not have an intuitive meaning. More meaningful are the uncertainties of the faulting angles (strike, dip, and rake shown in the main text), which result from the interaction of all six model parameters.

Although we used nonlinear SVD in this derivation, in practice we do not need to decompose $\mathbf{G}$. In fact, we only need to compute the eigenvalues and eigenvectors of $\mathbf{G}^T \mathbf{G}$. This shortcut further contributes to the numerical efficiency of the method. The parameter variances $\sigma^2(m_j)$ are actually the diagonal elements of $\mathbf{G}^T \mathbf{G}^{-1}$, and $\sigma^2(m_j)/\sigma_d^2$ are the diagonal elements of $\mathbf{G}^T \mathbf{G}^{-1}$. The singular vectors $\mathbf{V}_{(i)}$ of $\mathbf{G}$ are simply the eigenvectors of $\mathbf{G}^T \mathbf{G}$. The singular values of $\mathbf{G}$ can be calculated from the eigenvalues $\lambda_i$ of $\mathbf{G}^T \mathbf{G}$:

$$w_i = \sqrt{\lambda_i/\sigma_d^2}, \quad (A10)$$

and therefore,

$$\Delta \chi^2 = \frac{\lambda_1(V_{(1)} \cdot \delta \mathbf{m})^2 + \ldots + \lambda_6(V_{(6)} \cdot \delta \mathbf{m})^2}{\sigma_d^2}. \quad (A11)$$

If we now recall equations (7), (8), (9), and (A11), we find that the $\Delta \chi^2 = 1$ surface can be expressed in several equivalent ways:

![Figure A1. Two-dimensional cross-sections of 6D error ellipsoids along axes $m_1$ and $m_2$. Crosses, grid of points where $\Delta \chi^2$ is evaluated; diamonds, points for which $0 \leq \Delta \chi^2 \leq 1$. This example concerns the reference configuration described in Reference Setup: (a) 10-km-deep source, and (b) 40-km-deep source. The error ellipsoid of a 40-km-deep earthquake is larger than that of a 10-km-deep earthquake, indicating that $\mathbf{M}$ is better resolved for the 10-km-deep event. In this example the 6D-grid search evaluates $\sim 11^6$ points, from which $\sim 50,000$ fall inside the ellipsoid. The color version of this figure is available only in the electronic edition.](image-url)
\[ \sigma_d^2 = (\mathbf{d} - \mathbf{s}_{th})^2 - (\mathbf{d} - \mathbf{s}_{min})^2 = \sum_i d_i^2 (\text{VR}_{\text{min}} - \text{VR}_{\text{th}}) = \lambda_1 (V(1) \cdot \delta \mathbf{m})^2 + \ldots + \lambda_6 (V(6) \cdot \delta \mathbf{m})^2. \quad (A12) \]

Equation (A12) is an interesting by-product of our analysis: It demonstrates that choosing a family of acceptable solutions via prescription of a variance-reduction threshold (in case we have data) is equivalent to prescribing a value for data error \( \sigma_d \). Moreover, \( \sigma_d \) is explicitly related to the eigenvectors \( V(i) \) and \( \lambda_i \), which we can compute without data. This equivalence is not often recognized.

Let us next summarize the computational algorithm. Consider first that waveforms are available:

1. Matrix \( \mathbf{G} \) is formed; matrix \( \mathbf{G}^T \mathbf{G} \) is computed.
2. Column vector \( \mathbf{d} \) is formed.
3. The LSQ solution of (A4), given by (A5), provides the optimum solution \( \mathbf{m}_{\text{min}} \).
4. The LSQ solution is adopted as the reference solution \( \mathbf{m}_{\text{ref}} = \mathbf{m}_{\text{min}} \).
5. The singular vectors \( V(i) \) are computed as eigenvectors of \( \mathbf{G}^T \mathbf{G} \), and the singular values \( w_i \) are computed from the eigenvalues \( \lambda_i \) of \( \mathbf{G}^T \mathbf{G} \) according to (A10).
6. The standard deviations of the model parameters \( \sigma(\mathbf{m}) \) are obtained from (A8).
7. A 6D-grid search is performed to obtain the discrete points \( \delta \mathbf{m} \) satisfying \( 0 \leq \Delta \chi^2 \leq 1 \) (A9).
8. The set of model parameters \( \mathbf{m}_{\text{acceptable}} \) given by \( \mathbf{m}_{\text{acceptable}} = \mathbf{m}_{\text{ref}} + \delta \mathbf{m} \) is the family of acceptable solutions.

If waveform data are not available, and only the uncertainty analysis is to be made, we limit the analysis to steps 1 and 5–8. In this case, \( \mathbf{m}_{\text{ref}} \) is chosen arbitrarily. If we are interested in the resolvability of a specific focal mechanism, we can easily compute the coefficients \( m_j \) from strike, dip, rake, and scalar moment. We then proceed as indicated previously.

A final issue deserves attention: How does the uniform sampling of the 6D-parameter space relate to the sampling of the misfit \( \Delta \chi^2? \) Figure A2 shows the distribution of all values of \( \Delta \chi^2 \) found while grid-searching the 6D volume. The misfit \( \Delta \chi^2 \) is sufficiently smooth, with no holes or other artifacts due to the discretization of the parameter space. The distribution of \( \Delta \chi^2 \) is monotonous and strongly dominated by values of \( \Delta \chi^2 \) close to 1 (on the surface of the ellipsoid). In other words, due to the regular sampling of the parameter space, the focal-mechanism variability is primarily determined by the ellipsoid surface, not by its whole interior.

In fact, one would obtain identical distributions of the variability of the source parameters (strike, dip, rake, and the \( K \)-angle) whether using solutions inside the whole ellipsoid (0 \( \leq \Delta \chi^2 \leq 1 \)) or using solutions on the surface of the ellipsoid (e.g., 0.99 \( \leq \Delta \chi^2 \leq 1 \)). Thus, the focal-mechanism variability presented in this study (using 0 \( \leq \Delta \chi^2 \leq 1 \)) reflects the distribution of acceptable solutions not inside the whole ellipsoid, but rather on the ellipsoid surface.

Figure A2. Histogram of the misfit values inside the \( \Delta \chi^2 \leq 1 \) ellipsoid (found through 6D-grid search). Each parameter is varied by a maximum amount equal to plus or minus its standard deviation. This histogram concerns the reference setup described in Reference Setup (IB network, 40-km source depth). The histogram is completely independent of the actual focal mechanism, \( \mathbf{m}_{\text{ref}} \). The color version of this figure is available only in the electronic edition.

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