Initial-Value Approach for Viscoelastic Responses of the Earth’s Mantle

L. Hanyk¹, C. Matyska¹ and D.A. Yuen²

¹ Dept. of Geophysics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, CZ-180 00 Praha 8, Czech Republic
² Minnesota Supercomputer Institute and Dept. of Geology and Geophysics, University of Minnesota, Minneapolis, MN 55415-1227, USA

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Abstract. We have developed a theory based on the direct numerical integration in time for studying the temporal viscoelastic responses of earth models to surface loads. Modelling in the time domain is motivated by the fact that realistic elastically compressible models generate an infinite number of modes and the width of such “continuous spectrum” may cover several orders of magnitude in the Laplacian spectral domain for complicated viscosity stratification, which causes numerical difficulties for the normal-mode method. From our numerical solutions, we have directed our attention on the influences of elastic compressibility, thickness of the lithosphere, the nature of the internal mantle boundaries with density jumps and the viscosity structure near the interface between the lower and the upper mantle. There is a great difference between the responses of compressible and incompressible models mainly for shorter wavelengths and thus incompressible models seem to be inadequate for short-wavelength responses. The sensitivity of the viscoelastic responses to the lithospheric thickness in the presence of a low viscosity asthenosphere is substantial. This points to the need for constructing models with a 3-D viscosity variations, which would yield more realistic lithospheric-asthenospheric structure globally than models with a constant lithospheric thickness. The sensitivity of the Earth’s viscoelastic behaviour to the other parameters is weaker but still is noteworthy.

1 Introduction

The Earth’s mantle behaves as a viscoelastic solid for timescales ranging from one year to possibly millions of years. Viscoelastic models have been used to study the attenuation of the seismic normal modes [25, 45], earth tides [16, 32], postseismic rebound [23, 29, 1, 26], postglacial rebound (e.g. [24, 2, 44]), rotational dynamics [30, 46, 31, 28, 21, 12, 35, 20] and tectonics [37]. Various types of methods have been employed in solving the viscoelastic problems, ranging from normal modes [24], integral transform [29], finite elements [18, 33, 43] and initial-value techniques with spectral expansion in the spatial domain [2, 12, 10, 11]. There are distinct advantages for each of these methods, depending on the nature and spatial scale of the phenomenon being studied.

In the last several years there has been a resurgence of interest in the usage of initial-value techniques on integrating the set of ordinary differential equations obtained by spectral expansion of the partial differential equations of a pre-stressed, self-gravitating viscoelastic spherical model. The need for considering such an approach in contrast to the more traditional normal mode expansion using the Laplace transform [24] arises from the continuous spectrum found in stratified elastically compressible models [9] and continuous viscosity profiles [6, 7, 10, 11]. In
this work we will elaborate further on the differences between the initial-value and normal-mode approach, since there have been some issues raised recently [38, 39, 40].

We begin by discussing the mathematical formalism of this initial-value approach. We then follow by a discussion of the influences of the complex viscosity profiles on the temporal responses due to surface loads. The issues of compressibility versus incompressibility in the elastic regime and the influence of the thickness of the lithosphere will be discussed next. Next we delve into the role played by the nature of the boundary condition suggested by the transition zone’s phase changes on viscoelastic responses. Our final topic will deal with the effects of viscosity stratification under the transition zone on the viscoelastic responses.

2 Theory

2.1 Fundamental equations

The linear Maxwell viscoelastic rheology can be expressed as

$$\tau^M(t) = \tau^E(t) - \int_0^t \frac{\mu}{\eta}(\tau^M(t') - K \nabla \cdot \mathbf{u}(t') I) dt',$$

where

$$\tau^E = (K - \frac{2}{3}\mu)\nabla \cdot \mathbf{u} I + \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$$

is the elastic part of the “Maxwellian” stress tensor $\tau^M$; $I$ denotes the identity tensor, $\mathbf{u}$ is the displacement vector and $t$ is the time. Elastic properties of the Earth are described by the bulk modulus $K$ and the shear-stress modulus $\mu$, the viscous part of the rheology is represented by the dynamic viscosity $\eta$. The scalar product is denoted by $\cdot$ and the superscript $^T$ means the transpose.

The momentum equation of the pre-stressed self-gravitating continuum in a non-rotating reference system with inertial forces neglected is

$$\nabla \cdot \tau^M - \rho_0 \nabla \varphi_1 + \nabla \cdot (\rho_0 \mathbf{u}) \nabla \varphi_0 - \nabla (\rho_0 \nabla \cdot \varphi_0) = 0,$$

where $\varphi_0$ is the gravitational potential generated by the unperturbed density distribution $\rho_0$. Finally, the perturbation of the gravitational potential $\varphi_1$ satisfies the Poisson equation

$$\nabla \varphi_1 + 4\pi G \nabla \cdot (\rho_0 \mathbf{u}) = 0,$$

where $G$ is the gravitational constant.

2.2 Spherical harmonic decomposition

For density distribution and rheology parameters, which are spherically symmetric, i.e., $\rho_0 = \rho_0(r), K = K(r), \mu = \mu(r)$ and $\eta = \eta(r)$ with $r$ being the radial distance from the Earth’s centre, the system (1)–(4) can be decomposed by means of the spherical harmonic functions $Y_n(\Omega)$ representing any linear combination of the spherical harmonics $Y_{nm}(\theta, \varphi) = P_{nm}(\cos \theta) \exp(i m \varphi)$; here $P_{nm}$ are the associated Legendre functions [13] and $\Omega \equiv (\theta, \varphi)$ denotes the pair of angular coordinates formed by the colatitude $\theta$ and the longitude $\varphi$. Using $\mathbf{e}_r, \mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ as the unit basis vectors of the spherical coordinates, we can introduce the basis functions $\{Y_n(\Omega) \mathbf{e}_r, \nabla_\Omega Y_n(\Omega), \mathbf{e}_r \times \nabla_\Omega Y_n(\Omega)\}$ in the space of vector functions defined on a unit sphere, where

$$\nabla_\Omega Y_n(\Omega) \equiv \frac{\partial Y_n}{\partial \theta} \mathbf{e}_\theta + \frac{1}{\sin \theta} \frac{\partial Y_n}{\partial \varphi} \mathbf{e}_\varphi, \quad \mathbf{e}_r \times \nabla_\Omega Y_n(\Omega) \equiv -\frac{1}{\sin \theta} \frac{\partial Y_n}{\partial \varphi} \mathbf{e}_\theta + \frac{\partial Y_n}{\partial \theta} \mathbf{e}_\varphi.$$
The first and second component of this spherical harmonic vector function represent the basis function in the space of spheroidal vector functions whereas the last component is the basis function in the space of toroidal functions. We will use this basis in the following expansions

\[ u = \sum_n [U_n(\Omega)Y_n(\Omega)e_r + V_n(r)\nabla_\Omega Y_n(\Omega) + W_n(r)e_r \times \nabla_\Omega Y_n(\Omega)], \]

\[ e_r \cdot \mathbf{T}^E_M = \sum_n \left[ R_n^E(r)Y_n(\Omega)e_r + S_n^E(r)\nabla_\Omega Y_n(\Omega) + T_n^E(r)e_r \times \nabla_\Omega Y_n(\Omega) \right], \]

\[ e_r \cdot \int_0^t \mu(\mathbf{T}^E_M - K\nabla \cdot u I) dt' = \]

\[ \sum_n \left[ \alpha_n^E(r)Y_n(\Omega)e_r + \beta_n^E(r)\nabla_\Omega Y_n(\Omega) + \chi_n^E(r)e_r \times \nabla_\Omega Y_n(\Omega) \right], \]

\[ \nabla \cdot \int_0^t \mu(\mathbf{T}^E_M - K\nabla \cdot u I) dt' = \]

\[ \sum_n \left[ \gamma_n^E(r)Y_n(\Omega)e_r + \delta_n^E(r)\nabla_\Omega Y_n(\Omega) + \psi_n^E(r)e_r \times \nabla_\Omega Y_n(\Omega) \right], \]

where the symbol \( E|M \) means “either \( E \) or \( M \)”. Moreover, we will employ the expansion of \( \varphi_1 \) and \( \nabla \cdot u \) as follows,

\[ \varphi_1 = \sum_n F_n(r)Y_n(\Omega), \]

\[ \nabla \cdot u = \sum_n X_n(r)Y_n(\Omega), \quad X_n = \frac{d}{dr}U_n + \frac{2}{r}U_n - \frac{n(n+1)}{r}V_n. \]

2.3 Spheroidal part of the equations

We follow here the well-known fact from the theory of elasticity that the system of fundamental equations can be decoupled into its spheroidal and toroidal parts (see e.g. [8]). After introducing the vectors \( y_n^E, h_n, \)

\[ y_n^E(t) = (U_n, V_n, R_n^E, S_n^E, F_n, Q_n)^T, \]

\[ h_n(t) = (0, 0, \gamma_n^M, \delta_n^M, 0, 0)^T, \]

where \( Q_n = \frac{d}{dr}F_n + \frac{n+1}{r}F_n + 4\pi G\phi_0 U_n \), and the matrix \( A_n \) (see also [24])

\[ A_n = \begin{pmatrix}
-\frac{2\lambda}{r\beta} & \frac{N\lambda}{r\beta} & 1 & 0 & 0 & 0 \\
-\frac{1}{r} & \frac{1}{r} & \frac{1}{\beta} & 0 & 0 & 0 \\
\frac{4(\gamma - r\phi_0)}{r^2} & \frac{N(-2\gamma + r\phi_0)}{r^2} & -\frac{4\mu^\mu}{r} & -\frac{\phi_0(n+1)}{r} & \phi_0 & 0 \\
-\frac{2\mu + N(\gamma + \mu)}{r^2} & -\frac{2\mu + N(\gamma + \mu)}{r^2} & \frac{\lambda}{r\beta} & \frac{3}{r} & \phi_0 & 0 \\
-4\pi G\phi_0 & 0 & 0 & -\frac{n+1}{r} & 1 & 0 \\
\frac{4\pi G\phi_0(n+1)}{r} & \frac{4\pi G\phi_0N}{r} & 0 & 0 & 0 & \frac{n-1}{r}
\end{pmatrix} \]
where we have used the abbreviations $\beta = K + \frac{4}{3} \mu$, $\gamma = \frac{3\mu K}{\beta}$, $\lambda = K - \frac{2}{3} \mu$, $N = n(n+1)$, the spheroidal part of the system of fundamental equations (1)–(4) can be written as

$$\frac{d}{dt} y^E_n(t) = A_n y^E_n(t) + h_n(t).$$

Introducing the vector $y^M_n$,

$$y^M_n(t) = (U_n, V_n, R^M_n, S^M_n, F_n, Q_n)^T = y^E_n(t) - (0, 0, \alpha^M_n, \beta^M_n, 0, 0)^T,$$

we can write an analogous relation

$$\frac{d}{dt} y^M_n(t) = A_n y^M_n(t) + q^M_n(t).$$

For $q^M_n = h_n + \frac{d}{dt}(y^M_n - y^E_n) - A_n(y^M_n - y^E_n)$, it follows from (13), (14), (16) that

$$q^M_n = \begin{pmatrix}
\frac{\alpha^M_n}{\beta} \\
\frac{\beta^M_n}{\beta} \\
\gamma^M_n - \frac{d}{dt}\alpha^M_n - \frac{4\mu}{r\beta}\alpha^M_n + \frac{N}{r}\beta^M_n \\
\lambda^M_n - \frac{d}{dt}\beta^M_n - \frac{3\mu}{r\beta}\alpha^M_n - \frac{3}{r}\beta^M_n \\
0 \\
0
\end{pmatrix}.$$  \(18\)

These manipulations show that we must apply the same differential operator to $y^M_n$ as in the elastic problems. However, the system (17) of ordinary differential equations becomes now non-homogeneous. The vector $q^M_n$ represents the memory of the system. For performing the time-stepping of such a system, one needs to express $q^M_n$ by means of $y^M_n$ and to employ an appropriate integration scheme. These topics are discussed in Appendices 1 and 2, respectively.

### 2.4 Toroidal part of the equations

Following the derivation of the spheroidal part, we may introduce the vectors $z^{E|M}_n(t)$,

$$z^{E|M}_n(t) = (W_n, T^{E|M}_n)^T,$$  \(19\)

and the matrix $B_n$,

$$B_n = \begin{pmatrix}
1 & 1 \\
\frac{1}{r} & \frac{1}{r} \\
\mu(n-1)(n+2) & \frac{\mu}{r} \\
\frac{1}{r^2} & \frac{3}{r}
\end{pmatrix}.$$  \(20\)

The toroidal part of the fundamental equations (1)–(3) is then given by the system

$$\frac{d}{dt} z^M_n(t) = B_n z^M_n(t) + p^M_n(t),$$

where

$$p^M_n = \begin{pmatrix}
\frac{\chi^M_n}{\mu} \\
\psi^M_n - \frac{d}{dt}\chi^M_n - \frac{3}{r}\chi^M_n
\end{pmatrix}.$$  \(22\)

Detailed derivations of (22) as well as integration schemes are left again to Appendices 1 and 2.
2.5 Boundary conditions

In elastic problems, it is generally assumed that each interior boundary inside the mantle can be characterized by the continuity of \( \mathbf{u} \) and \( \mathbf{e}_r \cdot \mathbf{T}^E \). The core boundaries are described by the continuity of \( \mathbf{e}_r \cdot \mathbf{u} \) as well as \( \mathbf{e}_r \cdot \mathbf{T}^E \cdot \mathbf{e}_r \) and by the free-slip, i.e., \( \mathbf{e}_r \cdot \mathbf{T}^E - (\mathbf{e}_r \cdot \mathbf{T}^E \cdot \mathbf{e}_r)\mathbf{e}_r = 0 \). The boundary conditions inside the mantle mean that each inner boundary deforms together with boundary particles, i.e., there is no mass flux through the boundary.

In viscoelastic applications it is common to consider the same conditions as in elastic problems. However, if a boundary is caused by a phase transition, its position with respect to mantle particles can change. To satisfy conservation of mass, the continuity of \( \mathbf{e}_r \cdot \mathbf{u} \) then must be replaced by the continuity of \( \rho_0 \mathbf{e}_r \cdot \left( \frac{\partial \mathbf{u}}{\partial t} - \mathbf{v} \right) \), where \( \mathbf{v} \) is the velocity of the boundary motion. The conditions for horizontal components of displacement and for the traction \( \mathbf{e}_r \cdot \mathbf{T}^M \) remain unchanged in the linearized case because the conservation of momentum requires \( \rho_0 \mathbf{e}_r \cdot \left[ \frac{\partial \mathbf{u}}{\partial t} \left( \frac{\partial \mathbf{u}}{\partial t} - \mathbf{v} \right) - \mathbf{T}^M \right] \) to be continuous. In the general case, \( \mathbf{v} \) is a function of time. Therefore we can readily incorporate these boundary conditions only by direct time-integration of the viscoelastic responses.

At the surface we include the spectral representation of the Heaviside boundary condition for the point-mass load [8]:

\[
R^M_n = -g \Gamma_n , \quad S^M_n = T^M_n = 0 , \quad Q_n = -4\pi G \Gamma_n , \tag{23}
\]

where \( \Gamma_n = (2n + 1)/4\pi a^2 \) are the spectral coefficients of the Legendre series for the Dirac \( \delta \)-function.

3 Models and numerical results

The initial-value method has been proposed to be a direct numerical solver for computations of the temporal responses of viscoelastic Earth models with complicated viscosity profiles and realistic compressible elastic properties to all kinds of loading. Unless stated otherwise, we will employ the PREM model [5] for the density \( \rho_0(r) \), the bulk modulus \( K(r) \) and the shear-stress modulus \( \mu(r) \). To demonstrate complex viscoelastic response of the Earth, we start with the viscosity profile \( C \) (called C3 in [11]) that contains purely elastic lithosphere, low viscosity in the asthenosphere and a high viscosity “hill” in the lower mantle, consistent with the geodynamical modelling [27, 36] (Fig. 1).

![Fig. 1. The viscosity profile C with values spreading continuously over four orders in magnitude between the asthenosphere and the lower mantle. The elastic lithosphere is taken to be 120 km thick.](image)
Fig. 2. Time evolution of the surface values of the load Love numbers $h_n$, $l_n$ and $k_n$ ($n = 2, 6, 15, 60, 120$) for the PREM model and the viscosity profile C ($1 \text{ yr} < t < 1 \text{ Myr}$). Results for both the initial-value (solid curves) and the modal (dashed curves) approaches are plotted. Numbers of modes found by the modal solver for each $n$ are given at the right lower corners of the $h_n$-panels.
The time dependences of the Love numbers $h_n$ and $l_n$ for the Heaviside load obtained from explicit-like time integration scheme (see the case $\omega = 1$ in the appendix A.2) are drawn in Fig. 2. For comparing these results with the normal-mode technique employing the correspondence principle in the Laplace transform domain (e.g. [44, 9]), we have attempted to compute normal modes corresponding to the chosen model. The principal obstacle to be overcome lies in the fact that the number of modes tends to infinity and one needs a sophisticated numerical solver to be able to catch such weak modes that cannot be neglected a priori, as their sum may represent an observable part of the total response. We were able to find hundreds of modes for each angular order. However, when even a relatively large number of modes have been taken into account, we would still have difficulties in matching the curves obtained by direct time integration, as clearly seen in Fig. 2 for $n = 6$ and 60.

The existence of this "continuous spectrum" stems from these two facts. First, due to the continuity of the viscosity and/or the shear modulus variations the Laplacian shear modulus becomes singular at least at one depth for a singular bound of values of the Laplacian variable $s$ [7]. Because of the complexity of the viscosity profile, the singular bound of our model corresponds to the interval of relaxation times spanning from years to tens of kyr. To avoid the numerical difficulties with the propagator matrix in the Laplacian domain, we have made the continuous profiles of the shear modulus and viscosity discrete with a step corresponding to several kilometers. The singular bound of the discretized model is then a discrete set of values and we have scanned only the values of $s$ lying outside this discrete singular bound in a mode searching procedure. Second, the compressibility results in a generation of infinite number of weak modes ([9], see also [39]) and the question again arises whether they may be neglected. This sharp difference, occurring mainly in the horizontal displacement, was discussed by [19].

Fig. 3 shows the comparison between values of the Love numbers for elastically compressible and incompressible models at the time instants $t = 0$ (elastic response), 1 kyr, 10 kyr and infinity (the isostatic limit). The elastically compressible model has the bulk modulus $K(r)$ the same as the PREM, whereas the incompressible model uses the value of $K$ reaching infinity. The PREM values of the shear modulus and the viscosity profile drawn in Fig. 1 were considered in both cases. There is a fundamental difference between the elastically incompressible and compressible models at high angular orders because the incompressible models yield much higher attenuation of the Love numbers. We can clearly recognize that the relative difference in the Love number $h_n$ may exceed 100% for high angular orders. This fact would dictate against employing incompressible models, which are used mainly in regional studies.

However, even more important parameter is the thickness of the lithosphere, which can be much lower than 120 km (which is traditionally adopted in postglacial rebound modelling) in oceanic regions (e.g. [42, 22]). The values of the Love numbers of the compressible model $C$ for $t = 1$ kyr, 10 kyr and infinity and several lithosphere thicknesses (120, 100, 80 km) are shown in Fig. 4. One can clearly recognize that the response is faster with decreasing lithosphere thickness. Moreover, the amplitude of the whole spectrum of the Love number $h_n$ increases for thinner lithosphere. We can explain this in physical terms: a thinner lithosphere increases a role played by a low viscosity asthenosphere (the bottom of the asthenosphere was fixed at the depth of 220 km in each of the models). The thickness of the lithosphere can thus be a key parameter in viscoelastic modelling. Since the lithosphere properties are strongly laterally dependent mainly due to the difference between the continents and the oceans, our results point to an importance of developing more reliable methods capable of handling viscosities that could change several orders in the horizontal direction. Finite-element modelling, though effective for regional studies, may be hard-pressed in global modelling of 3-D viscosity structure for spherical geometries.

A fundamental problem of mantle dynamics is the nature of boundaries with large density
Fig. 3. Surface values of the load Love numbers $h_n$, $l_n$ and $k_n$ ($2 \leq n \leq 120$) for the PREM model and the viscosity profile C ($t = 0$, $1$, $10$, $\infty$ kyr). Solid and dashed curves represent the compressible and incompressible responses, respectively.

jumps and their relevant mathematical description. If a boundary is caused by a phase transition, it can potentially move with respect to surrounding particles during a dynamic process associated with changes of physical conditions. In elastic problems, such a mass flow through the boundary is prohibited because the time scale of elastic problems is very short in comparison with the time scale of phase transition kinetics, which ranges on the order between $10^3$ and $10^5$ years [4]. However, the time-scale of phase transition kinetics is comparable with the time-scale of viscoelastic responses, which should be taken into account in description of boundary conditions. To distinguish between the two end-member cases, we will denote a boundary without a mass flux as "chemical" and the one with a mass flux as the "phase" boundary. In
Fig. 4. Surface values of the load Love numbers $h_n$, $l_n$ and $k_n$ ($2 \leq n \leq 120$) for the PREM model at the time levels $t = 1$, 10 and $\infty$ kyr. The thickness of the lithosphere is 120, 100 or 80 km (solid, long-dashed or dashed curves, respectively). The asthenosphere with the bottom kept at the depth of 220 km is stretched upwards correspondingly, the viscosity profile $C$ is matched otherwise. The dotted curves stand for the elastic Love numbers ($t = 0$).

the case of the phase boundary, the radial component of velocity $\dot{U}_r$, where the dot denotes time derivative, cannot be continuous because of the conservation of mass (see Section 2.5).

In order to study the sensitivity of viscoelastic responses to the nature of the internal boundaries, we will here restrict ourselves only to the two limiting cases: i) the chemical boundaries with no mass flux and/or ii) the phase boundaries with $v = 0$, which correspond to boundaries with ultrafast kinetics immediately reaching their isostatic equilibrium position (see also [14]). The set of models taken into account can be described by a 4-index vector $(i_1, i_2, i_3, i_4)$, where the indices $i_1$, $i_2$ and $i_3$ correspond to the nature of the boundaries at the depths 400, 520 and 670 km respectively and may be equal to either the symbol $c$ meaning the chemical boundary or the symbol $p$ denoting the phase boundary. The density jumps in the depths 400 and 670 km are given by the PREM, densities around the depth 520 km were constructed by a slight modification of the PREM profile to get the same jump as that in the 400 km. Finally, the last index denotes one of the three viscosity profiles (see [44]): $L1$ is the model with constant viscosity $\eta = 10^{21}$ Pas, $L2$ has a viscosity jump by a factor of 10 across the 670 km discontinuity and the same value of upper-mantle viscosity as the $L1$ model, and the third model $L3$ has a low viscosity channel of $10^{19}$ Pas lying between 120 and 220 km depth, otherwise the same viscos-
Fig. 5. Percentual difference in the load Love numbers $h_n$ at the surface. The difference is taken between the predictions from models with all chemical boundaries (baseline case) and those with phase transitions as described in the text. Time is taken at $t = 20$ kyr and three viscosity profiles, L1, L2 and L3, have been employed.

Fig. 5 shows the relative changes in the surface values of the Love number $h_n$ at the time $t = 20$ kyr after the application of a Heaviside load if the models $(c, c, c, \{L1, L2, L3\})$ play the role of the reference models represented by the base level. The solid, dashed, dash-dotted and dotted curves represent respectively the responses from the models $(p, p, p, \{L1, L2, L3\})$, $(c, c, p, \{L1, L2, L3\})$, $(c, p, c, \{L1, L2, L3\})$, $(p, c, c, \{L1, L2, L3\})$. We can clearly recognize that the sensitivity of the Love number $h_n$ to the changes of the nature of the mantle internal boundaries may exceed 15% for this particular time. These relative changes are not very sensitive to a choice of the viscosity profile; only the presence of the low viscosity zone acts as a filter for suppressing the role of the boundaries for the higher angular orders. The influence of the boundary at 400 km depth is dominant for angular orders greater than 15, while the deepest boundary at 670 km depth is more influential for longer wavelengths. Inclusion of the phase boundary at 520 km [34] yields an additional contribution comparable to that at 400 km.

The time evolution of the Love numbers $h_2$, $l_2$ and $k_2$ for the models $(p, p, p, L1)$ (solid line) and $(c, c, c, L1)$ (dashed line) is compared in Fig. 6. A time necessary to reach hydrostatic equilibrium, when the two responses reach the same value, is very long (10 Myr). This value is comparable with characteristic times of mantle convection. This result points to the fact that a general viscoelastic response of the Earth’s mantle may be influenced not only by a time-

Fig. 6. Surface values of the load Love numbers $h_2$, $l_2$ and $k_2$ as a function of time. Solid and dashed curves denote respectively the phase transition and chemical boundary cases with three discontinuities at 400, 520 and 670 km depth. The viscosity profile L1 has been used.
Fig. 7. Depth variations of the radial ($R_2$) and tangential ($S_2$) spherical harmonic components of the stress tensor at the time $t = 20$ kyr. Solid and dashed curves denote the same models as in Fig. 6. The stress quantities have been non-dimensionalized.

evolution of a load but also by a time-evolution of model parameters, e.g. by an increase of viscosity in time, caused by tectonic evolution. We would like to emphasize that potential time-dependences of the model parameters would not require any change of our integration schemes.

Fig. 7 portrays the depth variations of the stress components $R_2^M$ and $S_2^M$ for the same models as in Fig. 6 and the time $t = 20$ kyr. The vertical traction is by two orders of magnitude higher than the horizontal one but the horizontal traction is very sensitive to the nature of the interior boundaries. While the model with the chemical boundaries yields maximal horizontal traction at the interface between the lower and upper mantle, the phase boundaries generate maximal horizontal traction at the base of the lithosphere.

Recently increased attention has been paid to the viscosity structure near the 670-km interface in geodynamical modelling (e.g., [15, 41]), which relates to the nature of the interfaces in the 670 and 1000 km depths. To study the sensitivity of the viscoelastic responses to potential viscosity changes at the top of the lower mantle, we will compare the results for the two models from Fig. 8. The model A is characterized by relatively high viscosity in the transition zone between the depths of 400 and 670 km and by a low viscosity layer in the upper part of the lower mantle. On the contrary, there is only a zone of lower viscosity just in the transition zone in the model B. The values of the Love numbers at the time instants $t = 0$, 1 kyr, 10 kyr and infinity are then shown in Fig. 9. We can see that the response of the model B is substantially faster. There is another interesting difference between the two models in the

Fig. 8. The viscosity profiles A, B are used in discussing effects of viscosity stratification near the transition zone on viscoelastic responses.
Fig. 9. The surface values of the load Love numbers $h_n$, $l_n$ and $k_n$ ($2 < n < 120$) for the PREM model and the viscosity profiles A (top), B (bottom). Dotted, dashed, long-dashed and solid curves show the time evolution ($t = 0, 1, 10, \infty$ kyr.)

Love numbers $h_n$ and $l_n$ for times of about several kyr: the peaks of the Love numbers for the model A are obtained for higher angular numbers than 30, which corresponds to the peak in the isostatic equilibrium, and, on the other side, the peaks of the model B are for lower angular numbers than 30. These results suggest that an effort to employ postglacial rebound data for constraining strongly depth-dependent viscosity models, used in geodynamical modelling might be profitable, provided we assume that the viscosities employed in the Maxwellian rheological models of the Earth’s viscoelastic response correspond to the viscosities employed in the geodynamical modelling.

4 Conclusions

There are five fundamental reasons for further development and the future increasing use of the initial-value technique in the modelling of viscoelastic responses:

1. The existence of the shear modulus singular bound in the Laplacian domain for models with a continuously changing profile [7].

2. An infinite number of modes for elastically compressible models [9].

3. The possibility to incorporate a time-dependent feedback (e.g. phase change kinetics) into the formulation of internal boundary conditions and structural time-dependences into the response modelling without a change of the integration scheme.

4. A possibility to incorporate general 3-D models of viscosity without any changes of the propagator matrix in the radial distance.

5. The increase in speed of raw computer power has made the time-marching scheme quite affordable. This scheme also allows for the code to be readily parallelized, thus further increasing the computational possibilities, especially for 3-D viscosity structure.

We have demonstrated that the viscoelastic responses of the mantle to surface loads are strongly sensitive to the lithospheric thickness, and thus models with realistic lithospheric and as-
thenospheric structures are highly desirable. The initial-value technique presented here can, in principle, be extended to models with an arbitrary 3-D viscosity. The differences between the compressible and incompressible models are especially acute for the higher angular degrees. Hence, employing elastically incompressible models for regional studies seems to be not of great value, in view of more accurate local geodetic measurements. The sensitivity of the viscoelastic response to the nature of the internal boundaries with density jumps and/or an order of magnitude viscosity jumps near the 670 km boundary are weaker but they can still exceed 10% for low angular orders.

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References


A Appendices

A.1 Derivation of the memory part of the equations

First, (17) yields the expression of \( \frac{d}{dr} U_n \) that may be put into (11). We obtain

\[
X_n = \frac{2\mu}{r^\beta} (2U_n - NV_n) + \frac{1}{\beta} R_n^M + \varphi_n^M.
\]  
(24)
Now, it is clear from eqs. (18) and (22) that $q^M_{n,1}$, $q^M_{n,2}$ and $p^M_{n,1}$ may be written as

\begin{align}
q^M_{n,1} &= \frac{1}{\beta} \int_0^t \frac{\mu}{\eta} (R^M_n - KX_n) \, dt'
\quad = \frac{1}{\beta} \int_0^t \frac{\mu}{\eta} \left[ \frac{2\gamma}{3r} (2U_n - NV_n) + \frac{4\mu}{3\beta} R^M_n - K\gamma^M_{n,1} \right] \, dt', \\
q^M_{n,2} &= \frac{1}{\mu} \int_0^t \frac{S^M_n}{\eta} \, dt', \\
p^M_{n,1} &= \frac{1}{\mu} \int_0^t \frac{\mu}{\eta} T^M_n \, dt'.
\end{align}

(25)

(26)

(27)

Derivation of the remaining terms is more complicated. If we use the relation (1) defining the kind of rheology, we may write analogous relations for the coefficients of the expansions (8) and (9):

\begin{align*}
\int_0^t \frac{\mu}{\eta} (\tau^M - K\nabla \cdot uI) \cdot e_\tau \, dt' &= \\
= \int_0^t \frac{\mu}{\eta} (\tau^E - K\nabla \cdot uI) \cdot e_\tau \, dt' - \int_0^t \frac{\mu}{\eta} \left( \int_0^t \frac{\mu}{\eta} (\tau^M - K\nabla \cdot uI) \cdot e_\tau \, dt'' \right) \, dt'
\end{align*}

(28)

and

\begin{align*}
\int_0^t \nabla \cdot \left[ \frac{\mu}{\eta} (\tau^M - K\nabla \cdot uI) \right] \, dt' &= \int_0^t \nabla \cdot \left[ \frac{\mu}{\eta} (\tau^E - K\nabla \cdot uI) \right] \, dt' - \\
&\quad - \int_0^t \frac{d}{dr} \left( \frac{\mu}{\eta} \right) \left( \int_0^t \frac{\mu}{\eta} (\tau^M - K\nabla \cdot uI) \cdot e_\tau \, dt'' \right) \, dt' - \int_0^t \frac{\mu}{\eta} \left( \int_0^t \nabla \cdot \left[ \frac{\mu}{\eta} (\tau^M - K\nabla \cdot uI) \right] \, dt'' \right) \, dt'
\end{align*}

(29)

In other words,

$$q^M_n = q^E_n - \int_0^t \frac{\mu}{\eta} q^M_n \, dt'.$$

(30)

Now we may use the expression of $\nabla \cdot \tau^E$ by means of the displacement vector in spherical coordinates to obtain

\begin{align*}
q^M_{n,3} &= \int_0^t \frac{\mu}{\eta} \frac{3}{r} (R^E - KX_n) \, dt' - \frac{4\mu}{r^3} \int_0^t \frac{\mu}{\eta} (R^E - KX_n) \, dt' - \int_0^t \frac{\mu}{\eta} q^M_{n,3} \, dt'
\quad = \frac{\gamma}{\mu r} \int_0^t \frac{\mu}{\eta} \left( -\frac{2\mu}{3} X_n + 2\mu \frac{d}{dr} U_n \right) \, dt' - \int_0^t \frac{\mu}{\eta} q^M_{n,3} \, dt'
\quad = \int_0^t \frac{\mu}{\eta} \left[ \frac{2\gamma}{r^2} \left( \frac{2\gamma}{3} X_n - 2U_n + NV_n \right) - q^M_{n,3} \right] \, dt'
\quad = \int_0^t \frac{\mu}{\eta} \left[ -\frac{2\gamma K}{r^2 \beta} (2U_n - NV_n) + \frac{4\gamma}{3r^2 \beta} R^M_n + \frac{4\gamma}{3r \mu q^M_{n,1} - q^M_{n,3}} \right] \, dt'.
\end{align*}

(31)

Analogously,

\begin{align*}
q^M_{n,4} &= \int_0^t \frac{\mu}{\eta} \frac{2}{r} \left( \frac{\partial Y_n}{\partial \theta} \right)^{-1} \left[ \frac{\partial}{\partial \theta} \left( -\frac{1}{3} X_n Y_n + \frac{1}{r} U_n Y_n + \frac{1}{r} V_n \frac{\partial^2 Y_n}{\partial \theta^2} \right) + \\
&\quad + \frac{1}{r \sin^2 \theta} V_n \left( \frac{\partial^2 Y_n}{\partial \theta \partial \varphi} - \cot \vartheta \frac{\partial^2 Y_n}{\partial \varphi^2} \right) + \cot \vartheta \frac{\partial}{r} V_n \left( \frac{\partial^2 Y_n}{\partial \theta^2} - \frac{\partial Y_n}{\partial \theta} \cot \vartheta - \frac{1}{\sin^2 \theta} \frac{\partial^2 Y_n}{\partial \varphi^2} \right) \right] \, dt' - \\
&\quad - \int_0^t \frac{\mu}{\eta} q^M_{n,4} \, dt'.
\end{align*}
\[- \frac{2 \lambda \mu}{r \beta} \int_0^t \frac{\mu}{\eta} \left( \frac{2}{3} X_n - \frac{2}{r} U_n + \frac{N}{r} V_n \right) dt' - \int_0^t \frac{\mu}{\eta} q_{n,4}^M dt'. \tag{32}\]

Since [13]

\[-N \frac{\partial}{\partial \theta} Y_n = \frac{\partial}{\partial \theta} \left( \frac{\partial^2 Y_n}{\partial \varphi^2} + \cot \vartheta \frac{\partial Y_n}{\partial \varphi} + \frac{1}{\sin^2 \vartheta} \frac{\partial^2 Y_n}{\partial \varphi^2} \right) = \]

\[\frac{\partial^3 Y_n}{\partial \varphi^3} + \cot \vartheta \frac{\partial^2 Y_n}{\partial \varphi^2} + \frac{1}{\sin^2 \vartheta} \frac{\partial^3 Y_n}{\partial \varphi^2} - 2 \cot \vartheta \frac{\partial^2 Y_n}{\sin^2 \vartheta} \frac{\partial Y_n}{\partial \varphi} - \left( \cot^2 \vartheta + 1 \right) \frac{\partial Y_n}{\partial \varphi} \tag{33}\]

we finally arrive at

\[q_{n,4}^M = \int_0^t \frac{\mu}{\eta} \left[ \frac{2 \gamma}{r^2} \left( -\frac{X_n}{3} + U_n \right) + \frac{2 \mu - N(\gamma + \mu)}{r^2} V_n - q_{n,4}^M \right] dt' \]

\[= \int_0^t \frac{\mu}{\eta} \left[ \frac{2 \gamma K}{r^2 \beta} U_n + \frac{2 \mu - N \left( \frac{2K}{r^2} + \mu \right)}{r^2} V_n - \frac{2 \gamma}{3r \beta} P_{n,1}^M - \frac{2 \gamma}{3r} q_{n,4}^M - q_{n,4}^M \right] \frac{\mu}{\eta} dt'. \tag{34}\]

The expressions (25), (26), (31) and (34) can be summarized as

\[q_n^M(t) = \int_0^t \frac{\mu}{\eta} \left[ \mathcal{Q}_n^M(t') + \mathcal{Q}_n q_n^M(t') \right] dt', \tag{35}\]

where

\[
\mathcal{Q}_n = \begin{pmatrix}
\frac{4 \gamma}{3r \beta} & \frac{2N \gamma}{3r \beta} & \frac{4 \mu}{3r \beta^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{4 \gamma K}{r^2 \beta} & \frac{2N \gamma K}{r^2 \beta} & \frac{4 \gamma}{3r \beta} & 0 & 0 & 0 \\
\frac{2 \gamma K}{r^2 \beta} & \frac{2 \mu - N \left( \frac{2K}{r^2} + \mu \right)}{r^2} & \frac{2 \gamma}{3r \beta} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \tilde{Q}_n = \begin{pmatrix}
\frac{K}{\beta} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3 \gamma}{3r} & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{36}\]

The same procedure applied to \(P_{n,2}^M\) yields

\[P_{n,2}^M = \int_0^t \frac{\mu}{\eta} \frac{r}{2} \left( \frac{\partial Y_n}{\partial \varphi} \right)^{-1} W_n \left[ \frac{\partial}{\partial \varphi} \left( \frac{\partial^2 Y_n}{\partial \varphi^2} - \frac{1}{\sin^2 \vartheta} \frac{\partial^2 Y_n}{\partial \varphi^2} \cot \vartheta \right) + \right] \]

\[+ \frac{2}{\sin^2 \vartheta} \left( \frac{\partial^2 Y_n}{\partial \vartheta \partial \varphi^2} - \cot \vartheta \frac{\partial^2 Y_n}{\partial \varphi^2} \right) + 2 \cot \vartheta \left( \frac{\partial^2 Y_n}{\partial \varphi^2} - \frac{\partial Y_n}{\partial \varphi} \right) \frac{1}{\sin^2 \vartheta} \frac{\partial^2 Y_n}{\partial \varphi^2} \right] dt' - \int_0^t \frac{\mu}{\eta} P_{n,2}^M dt' = - \int_0^t \frac{\mu}{\eta} \frac{r(N + 1)}{2} W_n dt' - \int_0^t \frac{\mu}{\eta} P_{n,2}^M dt'. \tag{37}\]

and thus the analogy of (35) and (36) is

\[p_n^M(t) = \int_0^t \frac{\mu}{\eta} \left[ \mathcal{P}_n z_n^M(t') + \mathcal{P}_n p_n^M(t') \right] dt', \tag{38}\]

where

\[
\mathcal{P}_n = \begin{pmatrix}
0 & \frac{1}{\mu} \\
\frac{\mu(N + 1)}{r^2} & 0
\end{pmatrix}, \quad \mathcal{P}_n = \begin{pmatrix}
0 & 0 \\
-1 & 0
\end{pmatrix}. \tag{39}\]
A.2 Time and spatial integration schemes

Let $t^0 = 0 < t^1 < \ldots < t^{i+1}$ be a series of successive time levels, let us denote $M^i = \frac{\mu}{\eta} (t^{i+1} - t^i)$, and let us consider the integration scheme for a function $f(t)$ in the form

$$\int_0^{t^{i+1}} \frac{\mu}{\eta} f(t) \, dt = \int_0^{t^i} \frac{\mu}{\eta} f(t) \, dt + M^i \left[ (\omega f^i + (1 - \omega) f^{i+1} ) \right],$$

with $0 \leq \omega \leq 1$, $f^i = f(t^i)$ and $f^{i+1} = f(t^{i+1})$. The discretized eqs. (17), (21) for $t = t^{i+1}$ read

$$\frac{d}{dr} y_{n}^{M,i+1} = A_n y_{n}^{M,i+1} + q_{n}^{M,i+1},$$

$$\frac{d}{dr} z_{n}^{M,i+1} = B_n z_{n}^{M,i+1} + p_{n}^{M,i+1}.$$

Applying (40) in (35), we get for $q_{n}^{M,i+1}$

$$q_{n}^{M,i+1} = q_{n}^{M,i} + M^i \left[ \omega (Q_n y_{n}^{M,i} + \tilde{Q}_n q_{n}^{M,i}) + (1 - \omega) (\tilde{Q}_n y_{n}^{M,i+1} + \tilde{Q}_n q_{n}^{M,i+1}) \right].$$

It yields

$$\left[ I - M^i (1 - \omega) \tilde{Q}_n \right] q_{n}^{M,i+1} = \left[ I + M^i \omega \tilde{Q}_n \right] q_{n}^{M,i} + M^i (1 - \omega) \tilde{Q}_n y_{n}^{M,i+1} + M^i \omega \tilde{Q}_n y_{n}^{M,i},$$

which—after substitution into (41)—gives the final expression,

$$\frac{d}{dr} y_{n}^{M,i+1} = \left[ A_n + M^i (1 - \omega) \tilde{A}_n \right] y_{n}^{M,i+1} + M^i \omega \tilde{A}_n y_{n}^{M,i} + \tilde{A}_n q_{n}^{M,i}.$$

We introduced

$$\tilde{A}_n = \left[ I - M^i (1 - \omega) \tilde{Q}_n \right]^{-1} \tilde{Q}_n,$$

$$\tilde{A}_n = \left[ I - M^i (1 - \omega) \tilde{Q}_n \right]^{-1} \left( I + M^i \omega \tilde{Q}_n \right),$$

where

$$\left[ I - M^i (1 - \omega) \tilde{Q}_n \right]^{-1} = \begin{pmatrix}
  c_3 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 \\
  2c_1 c_2 c_3 & 0 & c_2 & 0 & 0 \\
 -c_1 c_2 c_3 & 0 & 0 & c_2 & 0 \\
  0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

with $c_1 = \frac{2\gamma}{3r} M^i (1 - \omega)$, $c_2 = \frac{1}{1 + M^i (1 - \omega)}$ and $c_3 = \frac{1}{1 + M^i (1 - \omega) \frac{\eta}{\beta}}$. Note that the matrices $\tilde{A}_n$, $\tilde{A}_n$ are time-dependent only when the factors $M^i$ are time-dependent. The eq. (45) with the special choice of $\omega = 1$,

$$\frac{d}{dr} y_{n}^{M,i+1} = A_n y_{n}^{M,i+1} + M^i \tilde{Q}_n y_{n}^{M,i} + (I + M^i \tilde{Q}_n) q_{n}^{M,i},$$

was recently published by [11] in a slightly modified form.
The scheme for the toroidal part is

\[
\frac{d}{dr} \tilde{z}_{n,i+1} = \left[ B_{n} + M^i(1 - \omega) \tilde{B}_{n}^i \right] \tilde{z}_{n,i+1}^{M,i} + M^i \omega \tilde{B}_{n}^i \tilde{z}_{n,i}^{M,i} + \tilde{B}_{n}^i p_{n,i}^{M,i},
\]

where

\[
\tilde{B}_{n}^i = \begin{pmatrix}
0 & \frac{1}{\mu} \\
\frac{\mu(N + 1)}{r^2(1 + M^i(1 - \omega))} & 0
\end{pmatrix}, \quad \tilde{z}_{n}^i = \begin{pmatrix}
1 \\
0
\end{pmatrix}, \quad B_{n} = \begin{pmatrix}
1 & 0 \\
0 & 1 - M^i\omega
\end{pmatrix}.
\]

\[\tag{51}
\]

A.3 Extension of the theory to problems with 3-D viscosity

In a case with three-dimensional viscosity, it is necessary to work with the basis formed by the spherical harmonics \( Y_{nm} \), instead of harmonic functions \( Y_{n} \), i.e., the basis in the space of vector functions on a sphere is \( \{ Y_{nm}e_r, \nabla_\Omega Y_{nm}, e_r \times \nabla_\Omega Y_{nm} \} \). From the formal point of view, it is sufficient to replace the subscript \( n \) by \( nm \) in all formulas in the section 2. Note, however, that the propagator matrix \( A_{nm} \) is not dependent on \( m \) and is still defined by the relation (14). Let us take into account the following expansion

\[
\int_0^t \left( \frac{1}{r} \nabla_\Omega \left( \frac{\mu}{\eta} \right) + \frac{\mu}{\eta} \left( \nabla - \frac{\partial}{\partial r} e_r \right) \right) \cdot \left( \tau^M - K \nabla \cdot uI \right) dt' = \sum_{nm} [\bar{\gamma}_{nm} Y_{nm} e_r + \bar{\delta}_{nm} \nabla_\Omega Y_{nm} + \tilde{\psi}_{nm} e_r \times \nabla_\Omega Y_{nm}].
\]

\[\tag{52}
\]

The expression of the right-hand side of (18) can thus be replaced by

\[
\begin{pmatrix}
\alpha_{nm}^M \\
\beta_{nm}^M \\
\beta_{nm}^M
\end{pmatrix} =
\begin{pmatrix}
\bar{\gamma}_{nm} - \frac{4\mu}{r\beta} \alpha_{nm}^M + \frac{N}{r} \beta_{nm}^M \\
\bar{\delta}_{nm} - \frac{\lambda}{r\beta} \alpha_{nm}^M - \frac{3}{r} \beta_{nm}^M \\
0
\end{pmatrix}
\]

\[\tag{53}
\]

and (22) by

\[
p_{nm}^M = \begin{pmatrix}
\chi_{nm}^M \\
\mu \\
\bar{\psi}_{nm} - \frac{3}{r} \chi_{nm}^M
\end{pmatrix}.
\]

\[\tag{54}
\]

The explicit scheme of integration of the expansions can now be written as follows

\[
\sum_{nm} [\alpha_{nm}^{M,i+1} Y_{nm} e_r + \beta_{nm}^{M,i+1} \nabla_\Omega Y_{nm} + \chi_{nm}^{M,i+1} e_r \times \nabla_\Omega Y_{nm}] =
\]

\[
= \sum_{nm} [\alpha_{nm}^{M,i} Y_{nm} e_r + \beta_{nm}^{M,i} \nabla_\Omega Y_{nm} + \chi_{nm}^{M,i} e_r \times \nabla_\Omega Y_{nm}] + M^i(\tau^M,i - K \nabla \cdot u^iI) \cdot e_r,
\]

\[
\sum_{nm} [\bar{\gamma}_{nm}^{i+1} Y_{nm} e_r + \bar{\delta}_{nm}^{i+1} \nabla_\Omega Y_{nm} + \tilde{\psi}_{nm}^{i+1} e_r \times \nabla_\Omega Y_{nm}] =
\]
\[ = \sum_{nm} \left[ \tilde{\gamma}_{nm}^i Y_{nm} e_r + \tilde{\delta}_{nm} \nabla_r Y_{nm} + \tilde{\psi}_{nm}^i e_r \times \nabla_r Y_{nm} \right] + \\
 + \Delta t \left( \frac{1}{\tau} \nabla_\Omega \left( \frac{\mu}{\eta} \right) + \frac{\mu}{\eta} \left( \nabla \frac{\partial}{\partial r} e_r \right) \right) \cdot (\tau^{M,i} - K \nabla \cdot u^i I), \]  
\[ \text{(56)} \]

To be able to perform the time-stepping (55) and (56), we need to know the time evolution of the whole tensor \( \tau^M \). Let us denote \( \tau^V = - \int_0^{t_i} (\tau^M - K \nabla \cdot u^i I) \, dt' \). The integration scheme is

\[ \tau^{M,i} = \tau^{E,i} + \tau^{V,i-1} - \omega M^{i-1}(\tau^{M,i-1} - K \nabla \cdot u^{i-1} I) - (1 - \omega) M^{i-1}(\tau^{M,i} - K \nabla \cdot u^i I) \]  
\[ \text{(57)} \]

and thus

\[ \tau^{M,i} = \frac{1}{1 + (1 - \omega) M^i} \left[ \tau^{E,i} + (1 - \omega) M^{i-1} K \nabla \cdot u^i I + \tau^{V,i-1} - \omega M^{i-1}(\tau^{M,i-1} - K \nabla \cdot u^{i-1} I) \right] \]  
\[ \text{(58)} \]

Tensors \( \tau^{E,i} \) and \( \nabla \cdot u^i \) are linear combinations of \( U_{nm}^i, V_{nm}^i, W_{nm}^i, \frac{d}{dr} U_{nm}^i, \frac{d}{dr} V_{nm}^i, \frac{d}{dr} W_{nm}^i \) and due to (17) these quantities can be written as a linear combination of \( y_{nm}^i, \gamma_{nm}^i, q_{nm}^i, p_{nm}^i \). Hence, the time-stepping is, in principle, computable (see also [17]). Note here that the Laplace transform technique applied to problems with 3-D viscosity requires one to solve a coupled set of equations for each value of the Laplacian variable \( s \). [3] employed an iterative technique to solve these coupled equations for a model containing an axisymmetric craton with a modest increase in the lateral variations of the viscosity. The question arises as to the convergence of these iterations for a more general class of models, namely for more difficult models with sharp lateral variations in the viscosity involving several orders in magnitude.