Appendices

Appendix A

Scalar Spherical Harmonics

Fundamental features of scalar spherical harmonics are introduced. The relation between spherical harmonics and the associated Legendre functions and several other relations involving spherical harmonics are collected. The transform method for the evaluation of coefficients of spherical harmonic expansions by means of the fast Fourier transform and the Gauss-Legendre quadrature is discussed and presented on the particular case of a product of two scalar fields. Finally the transform method is applied to the evaluation of coefficients of a product of two zonal scalar fields expressed in terms of spherical harmonics or the derivatives of spherical harmonics.

A.1 Fundamental Features

The scalar spherical harmonics $Y_{nm}(\vartheta,\varphi)$ are unique, continuous and bounded complex functions of two real variables, colatitude ϑ and longitude φ, defined on the unit sphere, i.e., within the bounds of $0 \le \vartheta \le \pi$, $0 \le \varphi < 2\pi$; they are also referred to as the surface spherical harmonics. The two indices, degree n and order m (also angular order and azimuthal order, respectively), can take values of $n = 0, 1, 2, \ldots$ and $m = -n, -n+1, \ldots, n$. The following survey of features and expressions involving spherical harmonics is based on Jones (1985) and Varshalovich et al. (1988). Throughout this section, i denotes the imaginary unit and the asterisk stands for the complex conjugation.

The spherical harmonics $Y_{nm}(\vartheta,\varphi)$ are constructed to satisfy the two differential equations,

$$\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)+\frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}+n(n+1)\right]Y_{nm}(\vartheta,\varphi) = 0\,, \tag{A.1}$$

$$\left[i\frac{\partial}{\partial\varphi}+m\right]Y_{nm}(\vartheta,\varphi) = 0\,, \tag{A.2}$$

with the boundary conditions

$$Y_{nm}(\vartheta,0) = Y_{nm}(\vartheta,2\pi)\,, \tag{A.3}$$

$$\left.\frac{\partial}{\partial\varphi}Y_{nm}(\vartheta,\varphi)\right|_{\vartheta=0} = \left.\frac{\partial}{\partial\varphi}Y_{nm}(\vartheta,\varphi)\right|_{\vartheta=\pi} = 0\,. \tag{A.4}$$

The spherical harmonics are orthogonal for different n and m, and they are normalized so that

$$\int_0^{2\pi}\int_0^{\pi} Y_{nm}Y^*_{n'm'}\sin\vartheta\,d\vartheta\,d\varphi = \delta_{nn'}\delta_{mm'}\,, \tag{A.5}$$

while the phase is chosen to fulfil the condition

$$Y_{n0}(0,0) = \sqrt{\frac{2n+1}{4\pi}}\,. \tag{A.6}$$

As a consequence, several symmetries emerge, e.g.,

$$\begin{aligned} Y_{nm}(\vartheta,\varphi) &= (-1)^m Y^*_{n-m}(\vartheta,\varphi) &&= (-1)^m Y_{nm}(-\vartheta,\varphi) \\ &= (-1)^m Y_{n-m}(\vartheta,-\varphi) &&= Y_{n-m}(-\vartheta,-\varphi)\,, \end{aligned} \tag{A.7}$$

and the zonal spherical harmonics $Y_n(\vartheta)$,

$$Y_n(\vartheta) \equiv Y_{n0}(\vartheta)\,, \tag{A.8}$$

are both real for arbitrary values of ϑ and independent of φ. In Section 3.2, here and in Appendix B we adhere to the notation for the partial derivatives of $Y_{nm}(\vartheta,\varphi)$ and $Y_n(\vartheta)$ as follows:

$$\begin{aligned} Z_{nm}(\vartheta,\varphi) &= \frac{\partial Y_{nm}(\vartheta,\varphi)}{\partial\vartheta}\,, & \widetilde{Y}_{nm}(\vartheta,\varphi) &= \frac{1}{\sin\vartheta}\frac{\partial Y_{nm}(\vartheta,\varphi)}{\partial\varphi} &&= \frac{im\,Y_{nm}(\vartheta,\varphi)}{\sin\vartheta}\,, \\ Z_n(\vartheta) &= \frac{\partial Y_n(\vartheta)}{\partial\vartheta}\,, & \widetilde{Y}_n(\vartheta) &= \frac{1}{\sin\vartheta}\frac{\partial Y_n(\vartheta)}{\partial\varphi} &&= 0\,, \end{aligned} \tag{A.9}$$

where (A.2) has been substituted. All the partial derivatives of $Y_{nm}(\vartheta,\varphi)$ are also unique, continuous and bounded functions on the unit sphere. With r the radial distance from the center of the coordinate system r, ϑ, φ, it can be shown that functions $r^n Y_{nm}(\vartheta,\varphi)$ and $r^{-n-1}Y_{nm}(\vartheta,\varphi)$, referred to as the solid spherical harmonics, satisfy the Laplace equation,

$$\nabla^2 f(r,\vartheta,\varphi) = 0\,. \tag{A.10}$$

With $z_n(kr) = \sqrt{\frac{\pi}{2kr}}Z_{n+\frac{1}{2}}(kr)$ and $Z_n(kr)$ the spherical and the ordinary Bessel (or Neumann, or Hankel) functions of order n, respectively, it can be shown that functions $z_n(kr)Y_{nm}(\vartheta,\varphi)$ satisfy the homogeneous scalar Helmholtz equation with the constant eigenvalue k,

$$\left(\nabla^2 + k^2\right) f(r,\vartheta,\varphi) = 0\,. \tag{A.11}$$

Indeed, eq. (A.1) emerges during solution to these differential equations.

A.2 Connection with Associated Legendre Functions

The spherical harmonics $Y_{nm}(\vartheta,\varphi)$ are related to the associated Legendre functions $P_n^m(\cos\vartheta)$ by the equation applicable for $m \geq 0$, otherwise see (A.7),

$$Y_{nm}(\vartheta,\varphi) = (-1)^m N_{nm} P_n^m(\cos\vartheta)e^{im\varphi}\,, \qquad N_{nm} = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\,. \tag{A.12}$$

The associated Legendre functions $P_n^m(x)$, $x = \cos\vartheta$, can be defined in terms of the ordinary Legendre polynomials $P_n(x)$, which in turn can be introduced by the Rodrigues formula, respectively,

$$P_n^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P^n(x)\,, \tag{A.13}$$

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n\,. \tag{A.14}$$

Functions $P_n^m(x)$ satisfy numerous recursion relations, among them the following recurrence on n, supplemented by the closed-form expressions for the two starting values, is numerically stable,

$$(n-m)P_n^m(x) = (2n-1)xP_{n-1}^m(x) - (n+m-1)P_{n-2}^m(x)\,, \tag{A.15}$$

$$P_n^n(x) = (-1)^n \frac{(2n-1)!}{2^n n!}(1-x^2)^{\frac{n}{2}}\,, \tag{A.16}$$

$$P_{n+1}^n(x) = (2n+1)xP_n^n(x)\,. \tag{A.17}$$

For the zonal spherical harmonics, eq. (A.12) takes the simplified form

$$Y_n(\vartheta) = \sqrt{\frac{2n+1}{4\pi}}P_n(\cos\vartheta)\,. \tag{A.18}$$

The evaluation of the associated Legendre functions and the corresponding spherical harmonics of degrees $n \le 2$ leads to the following explicit expressions:

$$\begin{array}{lll}
P_0^0(x) = 1 & Y_{00} = \sqrt{\frac{1}{4\pi}} & Z_{00} = 0 \\
P_1^0(x) = x & Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\vartheta & Z_{10} = -\sqrt{\frac{3}{4\pi}}\sin\vartheta \\
P_1^1(x) = -\left(1-x^2\right)^{\frac{1}{2}} & Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\vartheta e^{\pm i\varphi} & Z_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}}\cos\vartheta e^{\pm i\varphi} \\
P_2^0(x) = \frac{3}{2}x^2 - \frac{1}{2} & Y_{20} = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\vartheta - 1\right) & Z_{20} = -\sqrt{\frac{45}{4\pi}}\sin\vartheta\cos\vartheta \\
P_2^1(x) = -3x\left(1-x^2\right)^{\frac{1}{2}} & Y_{2\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\vartheta\cos\vartheta e^{\pm i\varphi} & Z_{2\pm1} = \pm\sqrt{\frac{15}{8\pi}}\left(1-2\cos^2\vartheta\right)e^{\pm i\varphi} \\
P_2^2(x) = 3\left(1-x^2\right) & Y_{2\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\vartheta e^{\pm 2i\varphi} & Z_{2\pm2} = \sqrt{\frac{15}{8\pi}}\sin\vartheta\cos\vartheta e^{\pm 2i\varphi} \\
\text{and} & \cos\vartheta = \sqrt{\frac{4\pi}{3}}Y_{10} & \sin^2\vartheta = \frac{4}{3}\sqrt{\pi}\left(Y_{00} - \sqrt{\frac{1}{5}}Y_{20}\right)\,.
\end{array} \tag{A.19}$$

Numerous algebraic relations involving the spherical harmonics $Y_{nm}(\vartheta,\varphi)$ and $Y_n(\vartheta)$ and the derivatives $Z_{nm}(\vartheta,\varphi)$ and $Z_n(\vartheta)$ are available:

$$Z_{nm}\sin\vartheta = n\sqrt{\frac{(n+1)^2-m^2}{(2n+1)(2n+3)}}Y_{n+1,m} - (n+1)\sqrt{\frac{n^2-m^2}{(2n+1)(2n-1)}}Y_{n-1,m} \tag{A.20}$$

$$Z_n\sin\vartheta = \frac{n(n+1)}{\sqrt{(2n+1)(2n+3)}}Y_{n+1} - \frac{n(n+1)}{\sqrt{(2n+1)(2n-1)}}Y_{n-1} \tag{A.21}$$

$$Y_{nm}\cos\vartheta = \sqrt{\frac{(n-m+1)(n+m+1)}{(2n+1)(2n+3)}}Y_{n+1,m} + \sqrt{\frac{(n-m)(n+m)}{(2n-1)(2n+1)}}Y_{n-1,m} \tag{A.22}$$

$$Y_n\cos\vartheta = \frac{n+1}{\sqrt{(2n+1)(2n+3)}}Y_{n+1} + \frac{n}{\sqrt{(2n+1)(2n-1)}}Y_{n-1} \tag{A.23}$$

$$Z_n\cos\vartheta = \frac{n}{\sqrt{(2n+1)(2n+3)}}Z_{n+1} + \frac{n+1}{\sqrt{(2n+1)(2n-1)}}Z_{n-1} \tag{A.24}$$

$$Y_n\sin^2\vartheta = -\frac{(n+1)(n+2)Y_{n+2}}{(2n+3)\sqrt{(2n+1)(2n+5)}} + \frac{2(n^2+n-1)Y_n}{(2n-1)(2n+3)} - \frac{n(n-1)Y_{n-2}}{(2n-1)\sqrt{(2n-3)(2n+1)}} \tag{A.25}$$

$$Z_n\sin^2\vartheta = -\frac{n(n+1)Z_{n+2}}{(2n+3)\sqrt{(2n+1)(2n+5)}} + \frac{2n(n+1)Z_n}{(2n-1)(2n+3)} - \frac{n(n+1)Z_{n-2}}{(2n-1)\sqrt{(2n-3)(2n+1)}} \tag{A.26}$$

$$Z_n\sin\vartheta\cos\vartheta = \frac{n(n+1)(n+2)Y_{n+2}}{(2n+3)\sqrt{(2n+1)(2n+5)}} - \frac{n(n+1)Y_n}{(2n-1)(2n+3)} - \frac{(n-1)n(n+1)Y_{n-2}}{(2n-1)\sqrt{(2n-3)(2n+1)}}\,. \tag{A.27}$$

A.3 Spherical Harmonic Expansion of a Scalar Field

The spherical harmonics $Y_{nm}(\vartheta,\varphi)$ of integral degree and order, $n \geq 0$ and $|m| \leq n$, form a complete orthonormal basis of square-integrable functions of two real variables ϑ, φ on the unit sphere, $0 \leq \vartheta \leq \pi$, $0 \leq \varphi < 2\pi$. This means that for any function $f(\vartheta,\varphi)$ defined on the unit sphere and satisfying the condition

$$\int_0^{2\pi}\int_0^{\pi} |f(\vartheta,\varphi)|^2 \sin\vartheta\, d\vartheta\, d\varphi < \infty\,, \tag{A.28}$$

it is possible to find the spherical harmonic expansion of $f(\vartheta,\varphi)$,

$$f(\vartheta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_{nm} Y_{nm}(\vartheta,\varphi)\,, \tag{A.29}$$

where the coefficients f_{nm} can be expressed from (A.5),

$$f_{nm} = \int_0^{2\pi}\int_0^{\pi} f(\vartheta,\varphi) Y^*_{nm}(\vartheta,\varphi) \sin\vartheta\, d\vartheta\, d\varphi\,. \tag{A.30}$$

Condition (A.28) guarantees that

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n} |f_{nm}|^2 < \infty \qquad \text{and} \qquad \lim_{n\to\infty} |f_{nm}| = 0\,. \tag{A.31}$$

Formula (A.30) can be rewritten into the form of the Legendre-Fourier transform,

$$f_{nm} \quad = \quad \int_{-1}^{1}\left[\int_0^{2\pi} f(x,\varphi) e^{-im\varphi}\, d\varphi\right] (-1)^m N_{nm} P_n^m(x)\, dx\,, \tag{A.32}$$

with $x = \cos\vartheta$, which allows for the efficient evaluation using the fast Fourier transform and the Gauss-Legendre quadrature. The transform, or spectral, method based on the evaluation of (A.32) was first employed in the context of atmospheric modelling and, at present, it appears to be the regular textbook material (e.g., Dahlen & Tromp 1998, p. 943ff.). We apply it in the next two sections for the evaluation of coefficients of products of two scalar fields.

A.4 Spherical Harmonic Expansion of a Product of Two Scalar Fields

It is a common task to solve for coefficients of the spherical harmonic expansion of the product of two scalar fields, $h(\vartheta,\varphi) = f(\vartheta,\varphi)g(\vartheta,\varphi)$, given by the finite spherical harmonic expansions,

$$f(\vartheta,\varphi) \quad = \quad \sum_{n_1=0}^{N_1}\sum_{m_1=-n_1}^{n_1} f_{n_1m_1} Y_{n_1m_1}(\vartheta,\varphi)\,, \tag{A.33}$$

$$g(\vartheta,\varphi) \quad = \quad \sum_{n_2=0}^{N_2}\sum_{m_2=-n_2}^{n_2} g_{n_2m_2} Y_{n_2m_2}(\vartheta,\varphi)\,, \tag{A.34}$$

$$h(\vartheta,\varphi) \quad = \quad \sum_{n=0}^{N}\sum_{m=-n}^{n} h_{nm} Y_{nm}(\vartheta,\varphi)\,, \tag{A.35}$$

with the truncation degree of the expansion of $h(\vartheta,\varphi)$ bounded by $N \le N_1 + N_2$. Taking into account the Clebsch-Gordan expansion of the product of two spherical harmonics,

$$Y_{n_1m_1}(\vartheta,\varphi)Y_{n_2m_2}(\vartheta,\varphi) = \textstyle\sum_{nm} Q^{nm}_{n_1m_1n_2m_2} Y_{nm}(\vartheta,\varphi)\,, \tag{A.36}$$

$$Q^{nm}_{n_1m_1n_2m_2} = \sqrt{\frac{(2n_1+1)(2n_2+1)}{4\pi(2n+1)}}\, C^{n0}_{n_10n_20} C^{nm}_{n_1m_1n_2m_2}\,, \tag{A.37}$$

where $C^{nm}_{n_1m_1n_2m_2}$ are called the Clebsch-Gordan coefficients, the coefficients h_{nm} can be expressed by the sum

$$h_{nm} = \sum_{n_1m_1}\sum_{n_2m_2} f_{n_1m_1} g_{n_2m_2} Q^{nm}_{n_1m_1n_2m_2}\,. \tag{A.38}$$

However, it is more efficient, in terms of both computer time and memory, to evaluate the coefficients h_{nm} using the transform method based on (A.32). Here it takes the form of

$$h_{nm} = \int_0^{2\pi}\int_0^{\pi} h(\vartheta,\varphi) Y^*_{nm}(\vartheta,\varphi)\sin\vartheta\, d\vartheta\, d\varphi \tag{A.39}$$

$$= \int_{-1}^{1}\left[\int_0^{2\pi}\left(\sum_{n_1m_1} f_{n_1m_1}Y_{n_1m_1}(\vartheta,\varphi)\right)\left(\sum_{n_2m_2} g_{n_2m_2}Y_{n_2m_2}(\vartheta,\varphi)\right)e^{-im\varphi}\,d\varphi\right]\bar{Y}_{nm}(x)\,dx\,, \tag{A.40}$$

with $\bar{Y}_{nm}(x) \equiv (-1)^m N_{nm} P^m_n(\cos\vartheta)$ and $x = \cos\vartheta$. The expression in the brackets should be evaluated using the fast Fourier transform on the grid $\varphi_l = 2\pi l/L$, $l = 0,\ldots,L-1$. The rest can be accomplished by the Gauss-Legendre quadrature on the grid x_k, $k = 1,\ldots,K$. The grid points x_k should be equal to the roots of the Legendre polynomial $P_K(x)$, i.e., $P_K(x_k) = 0$. The sums in the parentheses should be evaluated for all the grid points $x_k = \cos\vartheta_k$ and φ_l. Hence, the following expressions are to be evaluated,

$$h(x_k,\varphi_l) = \left(\sum_{n_1=0}^{N_1}\sum_{m_1=-n_1}^{n_1} f_{n_1m_1}Y_{n_1m_1}(\vartheta_k,\varphi_l)\right)\left(\sum_{n_2=0}^{N_2}\sum_{m_2=-n_2}^{n_2} g_{n_2m_2}Y_{n_2m_2}(\vartheta_k,\varphi_l)\right) \tag{A.41}$$

$$\text{for } k = 1,\ldots,K,\quad l = 0,\ldots,L-1,$$

$$\bar{h}_m(x_k) = \frac{2\pi}{L}\sum_{l=0}^{L-1} h(x_k,\varphi_l)e^{-2\pi i m l/L} \qquad \text{for } |m| \le N,\quad k = 1,\ldots,K, \tag{A.42}$$

$$h_{nm} = \sum_{k=1}^{K} w_k^{(K)}\bar{h}_m(x_k)\bar{Y}_{nm}(x_k) \qquad \text{for } n = 0,\ldots,N,\quad |m| \le N\,, \tag{A.43}$$

where $w_k^{(K)}$ are the Gauss-Legendre weights. The choice of the values K, L is based on the chosen value of N. Generally, the coefficients h_{nm} are non-zero for any $n \le N = N_1 + N_2$, while the computed h_{nm}'s are often limited to $n \le N = N_1 = N_2$. In the former case, the choice of $L > N_1 + N_2 + N = 2(N_1 + N_2)$ is needed for the alias-free evaluation and L taken as a power of 2 is suitable for standard implementations of the fast Fourier transform; $K > (N_1 + N_2 + N)/2 = N_1 + N_2$ is required as the K-point Gauss-Legendre quadrature scheme ensures exact integration of degree $2K-1$ polynomials. In the latter case, the choice of $L > N_1 + N_2 + N = 3N$ (L is a power of 2) and $K > (N_1 + N_2 + N)/2 = \frac{3}{2}N$ is desirable. Note that the roots x_k of the Legendre polynomial $P_K(x)$ and the Gauss-Legendre weights $w_k^{(K)}$ are thought to be provided by any eligible routine, e.g., `gauleg` by Press et al. (1996).

For the zonal functions $f(\vartheta)$, $g(\vartheta)$ and $h(\vartheta) = f(\vartheta)g(\vartheta)$, with $Y_n(\vartheta) = \bar{Y}_n(x) = N_{n0}P_n(\cos\vartheta)$,

$$f(\vartheta) = \sum_{n_1=0}^{N_1} f_{n_1} Y_{n_1}(\vartheta)\,, \quad g(\vartheta) = \sum_{n_2=0}^{N_2} g_{n_2} Y_{n_2}(\vartheta)\,, \quad h(\vartheta) = \sum_{n=0}^{N} h_n Y_n(\vartheta)\,, \tag{A.44}$$

the transform method can be simplified to the evaluation of

$$h_n = 2\pi \int_{-1}^{1} h(x)\bar{Y}_n(x)\,dx = 2\pi \sum_{k=1}^{K} w_k^{(K)} \left(\sum_{n_1=0}^{N_1} f_{n_1} \bar{Y}_{n_1}(x_k) \right) \left(\sum_{n_2=0}^{N_2} g_{n_2} \bar{Y}_{n_2}(x_k) \right) \bar{Y}_n(x_k) \tag{A.45}$$

for $n = 0, \ldots, N$. Martinec (1989) implemented (A.40) as the routine `vcsum`, written in Fortran 77. For the zonal functions, we show the "one-statement" implementation of (A.45) using Fortran 90 in Appendix C.3.

A.5 Expansions in Terms of Derivatives of Spherical Harmonics

In Section 3.2 the coefficients of the spherical harmonic expansions of the products of two scalar axisymmetric (zonal) fields are presumed to be evaluated by the above outlined transform method. Specifically, products of the field $a(r,\vartheta)$, constant in time and prescribed a priori (viscosity, in particular), and of various fields $b(r,\vartheta)$, variable in time (e.g., displacement components), are to be evaluated. Fields $b(r,\vartheta)$ are given by the coefficients $b_n(r)$ of the two kinds of expansions, either $b(r,\vartheta) = \sum_{n=0}^{N} B_n(r)Y_n(\vartheta)$ or $b(r,\vartheta) = \sum_{n=1}^{N} B_n(r)Z_n(\vartheta)\sin\vartheta$, where $Z_n(\vartheta) = \partial Y_n(\vartheta)/\partial\vartheta$. Functions $Z_n(\vartheta)$ appear in the expansions accompanied with $\sin\vartheta$ since these products can be expressed as linear combinations of spherical harmonics $Y_{n-1}(\vartheta)$ and $Y_{n+1}(\vartheta)$, cf. (A.21). We remark that $Z_0(\vartheta)$ is zero identically.

The coefficients of the spherical harmonic expansions of the products $a(r,\vartheta)b(r,\vartheta)$, being also expressed as the linear combinations of either $Y_n(\vartheta)$ or $Z_n(\vartheta)\sin\vartheta$, will be denoted $\langle a; B_{n'}\rangle_{YY,n}$, $\langle a; B_{n'}\rangle_{ZY,n}$, $\langle a; B_{n'}\rangle_{YZ,n}$ and $\langle a; B_{n'}\rangle_{ZZ,n}$, in accordance with the following definitions,

$$\begin{aligned}
a(r,\vartheta) \textstyle\sum_{n=0}^{N} B_n(r)\, Y_n(\vartheta) &= \textstyle\sum_{n=0}^{\bar{N}} \langle a; B_{n'}\rangle_{YY,n}(r) Y_n(\vartheta)\,, \\
a(r,\vartheta) \textstyle\sum_{n=1}^{N} B_n(r)\, Z_n(\vartheta)\sin\vartheta &= \textstyle\sum_{n=0}^{\bar{N}} \langle a; B_{n'}\rangle_{ZY,n}(r) Y_n(\vartheta)\,, \\
a(r,\vartheta) \textstyle\sum_{n=0}^{N} B_n(r)\, Y_n(\vartheta) &= \textstyle\sum_{n=1}^{\bar{N}} \langle a; B_{n'}\rangle_{YZ,n}(r) Z_n(\vartheta)\sin\vartheta\,, \\
a(r,\vartheta) \textstyle\sum_{n=1}^{N} B_n(r)\, Z_n(\vartheta)\sin\vartheta &= \textstyle\sum_{n=1}^{\bar{N}} \langle a; B_{n'}\rangle_{ZZ,n}(r) Z_n(\vartheta)\sin\vartheta\,.
\end{aligned} \tag{A.46}$$

In general, the maximal truncation degree $\bar{N}$ of the first r.h.s. expansion equals to the sum of N and the truncation degree of the spherical harmonic expansion of $a(r,\vartheta)$. The truncation degree of the second r.h.s. expansion is one greater, as follows from (A.21). The truncation degrees of the third and the fourth r.h.s. expansions into $Y_n(\vartheta)$ would be identical with those of the first and the second r.h.s. expansions, respectively, but the expansions into $Z_n(\vartheta)\sin\vartheta$ can generally be infinite. However, we will presume that the spectral power of the constituents with $n > N$ is negligible, and $\bar{N} = N$ will be considered in the all cases.

We express the coefficients $\langle a; B_{n'}\rangle_{YY,n}$ and $\langle a; B_{n'}\rangle_{ZY,n}$ via the Gauss-Legendre quadrature, analogically to (A.45),

$$\begin{aligned}
\langle a; B_{n'} \rangle_{YY,n} &= 2\pi \int_{-1}^{1} a(r,x) b(r,x) \bar{Y}_n(x)\, dx \\
&= 2\pi \sum_{k=1}^{K} w_k^{(K)} a(r,x_k) \left[\sum_{n'=0}^{N} B_{n'}(r) \bar{Y}_{n'}(x_k) \right] \bar{Y}_n(x_k)\,, \qquad (A.47) \\
\langle a; B_{n'} \rangle_{ZY,n} &= 2\pi \int_{-1}^{1} a(r,x) b(r,x) \bar{Y}_n(x)\, dx \\
&= 2\pi \sum_{k=1}^{K} w_k^{(K)} a(r,x_k) \left[\sum_{n'=0}^{N+1} \left[\omega^+_{n'-1} B_{n'-1}(r) + \omega^-_{n'+1} B_{n'+1}(r) \right] \bar{Y}_{n'}(x_k) \right] \bar{Y}_n(x_k)\,, \qquad (A.48)
\end{aligned}$$

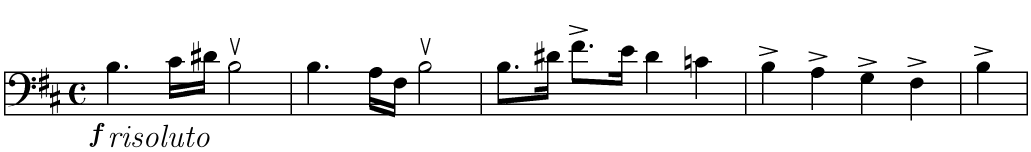
where relation (A.21), arranged according to

$$\sum_{n=1}^{N} B_n Z_n \sin\vartheta = \sum_{n=1}^{N} B_n \left(\omega_n^+ Y_{n+1} + \omega_n^- Y_{n-1} \right) = \sum_{n=0}^{N+1} \left(\omega^+_{n-1} B_{n-1} + \omega^-_{n+1} B_{n+1} \right) Y_n\,, \qquad (A.49)$$

has been involved; for coefficients ω_n^+ and ω_n^- see also (A.21). Applying similar reordering to the second pair of the r.h.s. expansions from (A.46), we arrive at the following tridiagonal systems of linear algebraic equations for $\langle a; B_{n'} \rangle_{YZ,n}$ and $\langle a; B_{n'} \rangle_{ZZ,n}$,

$$\begin{aligned}
\omega^+_{n-1} \langle a; B_{n'} \rangle_{YZ,n-1} + \omega^-_{n+1} \langle a; B_{n'} \rangle_{YZ,n+1} &= 2\pi \int_{-1}^{1} a(r,x) b(r,x) \bar{Y}_n(x)\, dx \\
&= 2\pi \sum_{k=1}^{K} w_k^{(K)} a(r,x_k) \left[\sum_{n'=0}^{N} B_{n'}(r) \bar{Y}_{n'}(x_k) \right] \bar{Y}_n(x_k)\,, \qquad (A.50) \\
\omega^+_{n-1} \langle a; B_{n'} \rangle_{ZZ,n-1} + \omega^-_{n+1} \langle a; B_{n'} \rangle_{ZZ,n+1} &= 2\pi \int_{-1}^{1} a(r,x) b(r,x) \bar{Y}_n(x)\, dx \\
&= 2\pi \sum_{k=1}^{K} w_k^{(K)} a(r,x_k) \left[\sum_{n'=0}^{N+1} \left[\omega^+_{n'-1} B_{n'-1}(r) + \omega^-_{n'+1} B_{n'+1}(r) \right] \bar{Y}_{n'}(x_k) \right] \bar{Y}_n(x_k)\,. \qquad (A.51)
\end{aligned}$$

We recall that $\langle a; B_{n'} \rangle_{YZ,0} = \langle a; B_{n'} \rangle_{ZZ,0} = 0$ and that N is the maximal truncation degree. By formal setting $\langle a; B_{n'} \rangle_{YZ,-1} = \langle a; B_{n'} \rangle_{ZZ,-1} = 0$, these two tridiagonal systems can be viewed as the recursion relations for $\langle a; B_{n'} \rangle_{YZ,n+1}$ and $\langle a; B_{n'} \rangle_{ZZ,n+1}$, with $n = 0, 1, \ldots, N-1$. Hence, we have derived the formulas (A.47)–(A.48) and (A.50)–(A.51) for the efficient evaluation of the coefficients introduced by (3.78) and (A.46).



ANTONÍN DVOŘÁK, *Concerto for Cello and Orchestra No. 2 in B minor, Op. 104. I. Allegro* (1895)