Appendix B

Scalar Representation of Vector and Tensor Fields

A subset of scalar representations of differential operators acting on scalar, vector and second-order tensor fields is derived. Concepts not necessarily needed for the purpose of this thesis are not involved; in particular, quantities are expressed in terms of physical rather than covariant and contravariant components. However, collected expressions form the apparatus which allows for the conversion of the field PDEs into the systems of ODEs and PDEs in Chapters 2 and 3. We begin from the formulas for the differential operators in orthogonal curvilinear coordinates (e.g., Morse & Feshbach 1953) and finish, essentially, with the expressions given by Backus (1967).

B.1 Differential Operators in Orthogonal Curvilinear Coordinates

Let $\xi_1$, $\xi_2$ and $\xi_3$ be the orthogonal curvilinear coordinates, $e_1$, $e_2$ and $e_3$ the corresponding unit basis vectors and $h_1$, $h_2$ and $h_3$ the Lamé coefficients defined by the relations to the Cartesian coordinates $x$, $y$ and $z$,

\[
h_i^2 = \left(\frac{\partial x}{\partial \xi_i}\right)^2 + \left(\frac{\partial y}{\partial \xi_i}\right)^2 + \left(\frac{\partial z}{\partial \xi_i}\right)^2, \quad H = h_1 h_2 h_3, \quad i = 1, 2, 3.
\]  

The differential operators \( \text{grad} \), \( \text{div} \) and \( \text{rot} \) acting on the scalar and vector functions

\[
f = f(\xi_1, \xi_2, \xi_3),
\]

\[
u = u(\xi_1, \xi_2, \xi_3) = u_1(\xi_1, \xi_2, \xi_3)e_1 + u_2(\xi_1, \xi_2, \xi_3)e_2 + u_3(\xi_1, \xi_2, \xi_3)e_3,
\]

can be introduced as follows (Morse & Feshbach 1953):

\footnote{Initial expressions of Section B.1 can be derived from definitions of differential operators given in terms of covariant derivatives. Expressions (B.1), (B.4)–(B.7) and (B.9) are retyped from Chapter 1 of the Russian translation of Morse & Feshbach (1953); however, we inverted the sign of the second r.h.s. term of (B.7). This correction is justifiable by derivation of (B.7) from the general definition of the $\nabla$ operator (not presented here), and also can be validated by comparing consequent expressions (B.8) and (B.26) with, e.g., Martinec (1989, p. 204) and Dahlen & Tromp (1998, p. 836), respectively.}

122
\[
\text{grad } f \equiv \nabla f = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} e_i = \frac{1}{h_i} \frac{\partial f}{\partial \xi_1} e_1 + \frac{1}{h_2} \frac{\partial f}{\partial \xi_2} e_2 + \frac{1}{h_3} \frac{\partial f}{\partial \xi_3} e_3, \quad (B.4)
\]

\[
\text{div } \mathbf{u} \equiv \nabla \cdot \mathbf{u} = \frac{1}{H} \sum_{i=1}^{3} \frac{\partial}{\partial \xi_i} \left( \frac{H u_i}{h_i} \right) = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial (h_3 h_2 u_1)}{\partial \xi_1} + \frac{\partial (h_1 h_3 u_2)}{\partial \xi_2} + \frac{\partial (h_1 h_2 u_3)}{\partial \xi_3} \right], \quad (B.5)
\]

\[
\text{rot } \mathbf{u} \equiv \nabla \times \mathbf{u} = \frac{1}{H} \sum_{ijkl} \epsilon_{ijkl} e_i \frac{\partial (h_k u_k)}{\partial \xi_j}, \quad (B.6)
\]

\[
\text{grad } \mathbf{u} = \left[ \frac{\partial}{\partial \xi_i} \left( \frac{u_i}{h_i} \right) + \mathbf{u} \cdot \nabla (\ln h_i) \right] \delta_{ij} + \frac{1}{2} \sum_{ij, i \neq j} \left[ \frac{h_{ij}}{h_i} \frac{\partial}{\partial \xi_j} \left( \frac{u_j}{h_j} \right) + \frac{h_{ij}}{h_j} \frac{\partial}{\partial \xi_i} \left( \frac{u_i}{h_i} \right) \right] e_i e_j, \quad (B.7)
\]

\[
\frac{1}{2} \left[ \nabla (\mathbf{u})^T + (\mathbf{u})^T \nabla \right] = \sum_{ij} \left[ \frac{\partial}{\partial \xi_i} \left( \frac{u_j}{h_j} \right) + \mathbf{u} \cdot \nabla (\ln h_i) \right] \delta_{ij} + \frac{1}{2} \sum_{ij, i \neq j} \left[ \frac{h_{ij}}{h_i} \frac{\partial}{\partial \xi_j} \left( \frac{u_j}{h_j} \right) + \frac{h_{ij}}{h_j} \frac{\partial}{\partial \xi_i} \left( \frac{u_i}{h_i} \right) \right] e_i e_j, \quad (B.8)
\]

with \( \delta_{kl} \) denoting the Kronecker symbol (\( \delta_{kl} = 1 \) for \( k = l \), otherwise 0) and \( \epsilon_{ijk} \) the Levi-Civita symbol (\( \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = -\epsilon_{132} = -\epsilon_{321} = 1 \), otherwise 0). Expressions for the second-order differential operators follow from (B.4)–(B.7); e.g., expression for the Laplace operator \( \Delta \equiv \text{div grad } \equiv \nabla^2 \equiv \nabla \cdot \nabla \) can be obtained by substitution of (B.4) into (B.5),

\[
\nabla^2 f = \frac{1}{H} \sum_{i=1}^{3} \frac{\partial}{\partial \xi_i} \left( \frac{H^2 \partial f}{h_i^2} \right) = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial (h_3 h_2 \partial f)}{\partial \xi_1} + \frac{\partial (h_1 h_3 \partial f)}{\partial \xi_2} + \frac{\partial (h_1 h_2 \partial f)}{\partial \xi_3} \right]. \quad (B.10)
\]

Similarly, expressions for the second-order operators

\[
\text{grad } \text{div } \mathbf{u} \equiv \nabla \nabla \cdot \mathbf{u}, \quad (B.11)
\]

\[
\text{rot } \text{rot } \mathbf{u} \equiv \nabla \times \nabla \times \mathbf{u}, \quad (B.12)
\]

follow from (B.4)–(B.5) and (B.7), respectively. Expressions involving tensor grad \( \mathbf{u} \),

\[
\text{div } \text{grad } \mathbf{u} \equiv \nabla^2 \mathbf{u}, \quad (B.13)
\]

\[
\text{div } (\text{grad } \mathbf{u})^T \equiv \nabla \cdot (\nabla \mathbf{u})^T, \quad (B.14)
\]

can be derived from (B.11), (B.12) and the relations

\[
\text{div } \text{grad } \mathbf{u} = \text{grad } \text{div } \mathbf{u} - \text{rot } \text{rot } \mathbf{u}, \quad \nabla^2 \mathbf{u} = \nabla \nabla \cdot \mathbf{u} - \nabla \times \nabla \times \mathbf{u}, \quad (B.15)
\]

\[
\text{div } (\text{grad } \mathbf{u})^T = \text{grad } \text{div } \mathbf{u}, \quad \nabla \cdot (\nabla \mathbf{u})^T = \nabla \nabla \cdot \mathbf{u}. \quad (B.16)
\]

The last two admissible second-order differential operators, \( \text{rot } \text{grad } \) and \( \text{div } \text{rot } \), are zero identically,

\[
\text{rot } f \equiv \nabla \times \nabla f = 0, \quad (B.17)
\]

\[
\text{div } \mathbf{u} \equiv \nabla \cdot \nabla \mathbf{u} = 0. \quad (B.18)
\]

We do not express the second-order differential operators in the orthogonal curvilinear coordinates explicitly. However, the spherical harmonic expansions of both the first-order (B.4)–(B.7) and the second-order (B.10)–(B.13) differential operators are given in Appendix B.3.
B.2 Differential Operators in Spherical Coordinates

Let $\xi_1 = r$, $\xi_2 = \vartheta$ and $\xi_3 = \varphi$ be the spherical coordinates defined by the relations to the Cartesian coordinates $x$, $y$ and $z$,

$$x = r \sin \vartheta \cos \varphi, \quad y = r \sin \vartheta \sin \varphi, \quad z = r \cos \vartheta,$$

with $e_r$, $e_\vartheta$ and $e_\varphi$ the unit basis vectors. The Lämé coefficients of the spherical coordinates are equal to

$$h_r = 1, \quad h_\vartheta = r, \quad h_\varphi = r \sin \vartheta.$$

For the scalar and vector functions

$$f = f(r, \vartheta, \varphi),$$

$$u = u(r, \vartheta, \varphi) = u_r(r, \vartheta, \varphi)e_r + u_\vartheta(r, \vartheta, \varphi)e_\vartheta + u_\varphi(r, \vartheta, \varphi)e_\varphi,$$

expressions (B.4)–(B.7) take the specific form:

$$\nabla f = \frac{\partial f}{\partial r} e_r + \frac{1}{r} \nabla_\Omega f = \frac{\partial f}{\partial r} e_r + \frac{1}{r \sin \vartheta} \frac{\partial f}{\partial \vartheta} e_\vartheta + \frac{1}{r \sin \vartheta \sin \varphi} \frac{\partial f}{\partial \varphi} e_\varphi,$$

$$\nabla \cdot u = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \vartheta} (u_\vartheta \sin \vartheta) + \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \varphi} u_\varphi,$$

$$\nabla \times u = \frac{1}{r \sin \vartheta} \left[ \frac{\partial (u_\varphi \sin \vartheta)}{\partial \vartheta} - \frac{\partial u_\vartheta}{\partial \varphi} \right] e_r + \frac{1}{r} \left[ \frac{1}{\sin \vartheta} \frac{\partial u_r}{\partial \varphi} - \frac{\partial (ru_\varphi)}{\partial r} \right] e_\vartheta + \frac{1}{r} \left[ \frac{\partial (ru_\varphi)}{\partial r} - \frac{\partial u_r}{\partial \varphi} \right] e_\varphi,$$

$$\nabla u = \sum_{k=1}^3 \sum_{l=1}^3 \left[ (\nabla u)_{kl} e_k e_l \right],$$

$$(\nabla u)_{rr} = \frac{\partial u_r}{\partial r}, \quad (\nabla u)_{r\vartheta} = \frac{1}{r} \frac{\partial u_r}{\partial \vartheta} - \frac{u_\vartheta}{r}, \quad (\nabla u)_{r\varphi} = \frac{1}{r \sin \vartheta \sin \varphi} \frac{\partial u_r}{\partial \varphi} - \frac{u_\varphi}{r},$$

$$(\nabla u)_{\vartheta r} = \frac{\partial u_\vartheta}{\partial r}, \quad (\nabla u)_{\vartheta\vartheta} = \frac{1}{r} \frac{\partial u_\vartheta}{\partial \vartheta} + \frac{u_r}{r}, \quad (\nabla u)_{\vartheta\varphi} = \frac{1}{r \sin \vartheta} \frac{\partial u_\vartheta}{\partial \varphi} + \frac{u_\varphi}{r},$$

$$(\nabla u)_{\varphi r} = \frac{\partial u_\varphi}{\partial r}, \quad (\nabla u)_{\varphi\vartheta} = \frac{1}{r \sin \vartheta} \frac{\partial u_\varphi}{\partial \vartheta}, \quad (\nabla u)_{\varphi\varphi} = \frac{1}{r} \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\vartheta}{r} + \frac{u_\vartheta \cos \varphi}{r \sin \vartheta}.$$

By $\nabla_\Omega = e_\vartheta \frac{\partial}{\partial \vartheta} + e_\varphi \frac{1}{\sin \vartheta} \frac{\partial}{\partial \varphi}$ we denote the angular part of the $\nabla$ operator, the surface gradient. From the second-order differential operators, expression for the Laplace operator (B.10) is only rewritten here:

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial f}{\partial \vartheta} \right) + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 f}{\partial \varphi^2}.$$

B.3 Differential Operators Acting on Spherical Harmonic Expansions

Let $r$, $\vartheta$ and $\varphi$ be the spherical coordinates related to the Cartesian coordinates $x$, $y$ and $z$ by (B.19), $e_r$, $e_\vartheta$ and $e_\varphi$ the corresponding unit basis vectors, $Y_{nm}(\vartheta, \varphi)$ the scalar spherical harmonics (A.12) and $Z_{nm}(\vartheta, \varphi)$ and $Y_{nm}(\vartheta, \varphi)$ the partial derivatives of $Y_{nm}(\vartheta, \varphi)$ defined by (A.9). Let $S_{nm}^{(1)}$, $S_{nm}^{(2)}$ and $S_{nm}^{(0)}$ be the vector spherical harmonics,
The spherical harmonic expansions of scalar and vector functions can be expressed as
\[ f(r, \vartheta, \varphi) = \sum_{nm} F_{nm}(r) Y_{nm}(\vartheta, \varphi), \] (B.31) and
\[ \mathbf{u}(r, \vartheta, \varphi) = \sum_{nm} \left[ U_{nm}(r) S_{nm}^{(-1)}(\vartheta, \varphi) + V_{nm}(r) S_{nm}^{(1)}(\vartheta, \varphi) + W_{nm}(r) S_{nm}^{(0)}(\vartheta, \varphi) \right]. \] (B.32)

Expressions for the scalar components of \( \mathbf{u}(r, \vartheta, \varphi) \) in the directions of \( \mathbf{e}_r, \mathbf{e}_\vartheta \) and \( \mathbf{e}_\varphi \) follow from substitution of the vector spherical harmonics (B.28)–(B.30) into (B.32),
\[ u_r(r, \vartheta, \varphi) = \sum_{nm} \left[ U_{nm}(r) Y_{nm}(\vartheta, \varphi) \right], \] (B.33)
\[ u_\vartheta(r, \vartheta, \varphi) = \sum_{nm} \left[ V_{nm}(r) Z_{nm}(\vartheta, \varphi) - W_{nm} \tilde{Y}_{nm}(\vartheta, \varphi) \right], \] (B.34)
\[ u_\varphi(r, \vartheta, \varphi) = \sum_{nm} \left[ V_{nm}(r) \tilde{Y}_{nm}(\vartheta, \varphi) + W_{nm} Z_{nm}(\vartheta, \varphi) \right]. \] (B.35)

Expressions (B.23)–(B.26) for the first-order differential operators acting on the spherical harmonic expansions (B.31)–(B.32) can be rewritten using (B.33)–(B.35) into the form
\[ \nabla f = \sum_{nm} \left( F_{nm} S_{nm}^{(-1)} + \frac{F_{nm}}{r} S_{nm}^{(1)} \right) Y_{nm}, \] (B.36)
\[ \nabla \cdot \mathbf{u} = \sum_{nm} \left( U_{nm} + \frac{2U_{nm} - NV_{nm}}{r} \right) Y_{nm}, \] (B.37)
\[ \nabla \times \mathbf{u} = \sum_{nm} \left[ -\frac{NW_{nm}}{r} S_{nm}^{(-1)} - \left( W_{nm} + \frac{W_{nm}}{r} \right) S_{nm}^{(1)} + \left( V_{nm} + \frac{V_{nm} - U_{nm}}{r} \right) S_{nm}^{(0)} \right], \] (B.38)
where \( N = n(n+1) \) and the prime ' stands for the derivative with respect to \( r \). From now on we suppress the degree and order subscripts of the coefficients \( F_{nm}, U_{nm}, V_{nm} \) and \( W_{nm} \). The second-order differential operators take the form as follows:
\[ \text{div} \ \text{grad} \ f = \sum_{nm} \left( F_{nm} + \frac{2F_{nm}'}{r^2} - \frac{NF_{nm}'}{r^2} \right) Y_{nm}, \] (B.39)
\[ \text{grad} \ \text{div} \ \mathbf{u} = \sum_{nm} \left\{ \left( U' + \frac{2U - NV}{r} \right) S_{nm}^{(-1)} + \frac{1}{r} \left( U' + \frac{2U - NV}{r} \right) S_{nm}^{(1)} \right\}, \] (B.40)
\[ \text{rot} \ \text{rot} \ \mathbf{u} = \sum_{nm} \left\{ -\frac{N}{r} \left( V' + \frac{V - U}{r} \right) S_{nm}^{(-1)} - \left( V' + \frac{V - U}{r} \right)' + \frac{1}{r} \left( V' + \frac{V - U}{r} \right) \right\} S_{nm}^{(1)} \]
\[-\left( W' + \frac{W}{r} \right)' + \frac{1}{r} \left( W' + \frac{W}{r} \right) - \frac{NW_{nm}}{r^2} \right\} S_{nm}^{(0)} \right\}, \] (B.41)
\[ \text{div} \ \text{grad} \ \mathbf{u} = \sum_{nm} \left\{ \left( U' + \frac{2U - NV}{r} \right)' + \frac{N}{r} \left( V' + \frac{V - U}{r} \right) \right\} S_{nm}^{(-1)} \]. (B.42)
\[
+ \left[ \frac{1}{r} \left( U' + \frac{2U - NV}{r} \right) + \left( V' + \frac{V - U}{r} \right)' + \frac{1}{r} \left( V' + \frac{V - U}{r} \right) \right] S_{nm}^{(1)}
\]

\[
+ \left( W' + \frac{W}{r} \right)' + \frac{1}{r} \left( W' + \frac{W}{r} \right) - \frac{NW}{r^2} \right] S_{nm}^{(0)} \right)
\]

\[
= \sum_{nm} \left[ U'' + \frac{2U'}{r} - \frac{(N + 2)U}{r^2} + \frac{2NV}{r^2} \right] S_{nm}^{(-1)} + \left( V'' + \frac{2V'}{r} + \frac{2U - NV}{r^2} \right) S_{nm}^{(1)}
\]

\[
+ \left( W'' + \frac{2W'}{r} - \frac{NW}{r^2} \right) S_{nm}^{(0)} \right].
\]

Expressions (B.39) and (B.40) follow from (B.36)–(B.37), expression (B.41) is obtained from (B.38) and expression (B.42) from (B.15). Two relations involving the scalar spherical harmonics have also been used,

\[
\frac{\partial Z_{nm}}{\partial \vartheta} + \frac{\cos \vartheta}{\sin \vartheta} Z_{nm} + \frac{1}{\sin \vartheta} \frac{\partial \tilde{Y}_{nm}}{\partial \varphi} = -NY_{nm},
\]

\[
\frac{\partial \tilde{Y}_{nm}}{\partial \vartheta} + \frac{\cos \vartheta}{\sin \vartheta} \tilde{Y}_{nm} - \frac{1}{\sin \vartheta} \frac{\partial Z_{nm}}{\partial \varphi} = 0,
\]

the former being (A.1) rewritten in terms of \(Z_{nm}\) and \(\tilde{Y}_{nm}\) and the latter following from the definition of \(\tilde{Y}_{nm}\) in (A.9) by differentiation with respect to \(\vartheta\). Finally, we supply the selected expressions for the differential operators acting on components of \(u\) in the basis \(S_{nm}^{(-1)}\), \(S_{nm}^{(1)}\) and \(S_{nm}^{(0)}\). Specifically, the div operator,

\[
\nabla \cdot U S_{nm}^{(-1)} = (U' + 2U/r) Y_{nm},
\]

\[
\nabla \cdot V S_{nm}^{(1)} = -(NV/r) Y_{nm},
\]

\[
\nabla \cdot W S_{nm}^{(0)} = 0,
\]

the rot operator,

\[
\nabla \times U S_{nm}^{(-1)} = -(U/r) S_{nm}^{(0)},
\]

\[
\nabla \times V S_{nm}^{(1)} = (V' + V/r) S_{nm}^{(0)},
\]

\[
\nabla \times W S_{nm}^{(0)} = -(NW/r) S_{nm}^{(-1)} - (W' + W/r) S_{nm}^{(1)},
\]

the grad div operator,

\[
\nabla \nabla \cdot U S_{nm}^{(-1)} = (U' + 2U/r)' S_{nm}^{(-1)} + (U' + 2U/r^2) S_{nm}^{(1)},
\]

\[
\nabla \nabla \cdot V S_{nm}^{(1)} = -(NV/r)' S_{nm}^{(-1)} - (NV/r^2) S_{nm}^{(1)},
\]

\[
\nabla \nabla \cdot W S_{nm}^{(0)} = 0,
\]

and the rot rot operator are provided,

\[
\nabla \times \nabla \times U S_{nm}^{(-1)} = (NU/r^2) S_{nm}^{(-1)} + (U'/r) S_{nm}^{(1)},
\]

\[
\nabla \times \nabla \times V S_{nm}^{(1)} = -(NV'/r + NV/r^2) S_{nm}^{(-1)} - (V'' + 2V'/r) S_{nm}^{(1)},
\]

\[
\nabla \times \nabla \times W S_{nm}^{(0)} = -(W'' + 2W'/r - NW/r^2) S_{nm}^{(0)}.
\]
B.4 Strain Tensor

The scalar components of the symmetric strain tensor \( e \), defined by
\[
e = \frac{1}{2} \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right],
\]
can be expressed in the spherical coordinates using the components of \( \nabla \mathbf{u} \) by (B.26), and the spherical harmonic expansions of these components can be obtained by the subsequent substitution from (B.33)–(B.35) as follows,
\[
\begin{align*}
\begin{pmatrix}
\epsilon_{rr} & 2\epsilon_{r\theta} & 2\epsilon_{r\varphi} \\
\epsilon_{\theta\theta} & 2\epsilon_{\theta\varphi} \\
\epsilon_{\varphi\varphi}
\end{pmatrix} &=
\begin{pmatrix}
\frac{\partial u_r}{\partial r} & \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{u_\theta}{r} & \frac{1}{r} \frac{\partial u_r}{\partial \varphi} + \frac{u_\varphi}{r} \\
\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} & \frac{1}{r} \frac{\partial u_\theta}{\partial \varphi} + \frac{u_\varphi}{r} & \frac{1}{r} \frac{\partial u_\theta}{\partial \varphi} \\
\frac{1}{r} \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r}{r} & \frac{1}{r} \frac{\partial u_\varphi}{\partial \varphi} & \frac{1}{r} \frac{\partial u_\varphi}{\partial \varphi}
\end{pmatrix}
\end{align*}
\]
\[= \sum_{nm} \left( U' Y_{nm} \left( V' + \frac{U - V}{r} \right) Z_{nm} - \left( W' - \frac{W}{r} \right) \bar{Y}_{nm} \right) \left( V' + \frac{U - V}{r} \right) \bar{Y}_{nm} + \left( W' - \frac{W}{r} \right) Z_{nm} \right)
\]
\[
\begin{align*}
\sum_{nm} \left( \frac{U}{r} Y_{nm} + \frac{V}{r} \frac{\partial Z_{nm}}{\partial \theta} - \frac{W}{r} \frac{\partial Y_{nm}}{\partial \theta} \right)
\end{align*}
\]
\[
\sum_{nm} \left( \frac{U}{r} Y_{nm} - \frac{V}{r} \frac{\partial Z_{nm}}{\partial \theta} + \frac{W}{r} \frac{\partial Y_{nm}}{\partial \theta} \right)
\]
\[
\sum_{nm} \left( \frac{U}{r} Y_{nm} \right)
\]
Symmetry of the tensors is indicated by the double quotes. From (B.51), the first invariant of \( e \) can be found,
\[
e \equiv e_{rr} + e_{\theta\theta} + e_{\varphi\varphi} = \sum_{nm} \left( U' + \frac{2U - NV}{r} \right) Y_{nm} = \nabla \cdot \mathbf{u}.
\]
We express the spherical harmonic expansions of the vectors \( e_r \cdot e, e_\theta \cdot e \) and \( e_\varphi \cdot e \),
\[
e_r \cdot e = e_r e_r + e_\theta e_\theta + e_\varphi e_\varphi
\]
\[
e_r = \sum_{nm} \left[ \frac{1}{2} \left( V' + \frac{U - V}{r} \right) S_{nm}^{(1)} + \frac{1}{2} \left( W' - \frac{W}{r} \right) S_{nm}^{(0)} \right],
\]
\[
e_\theta = \sum_{nm} \left[ \frac{1}{2} \left( V' - \frac{U - V}{r} \right) \frac{\partial S_{nm}^{(1)}}{\partial \theta} + \frac{W}{r} \left( \frac{\partial S_{nm}^{(0)}}{\partial \theta} + N Y_{nm} e_\varphi \right) \right],
\]
\[
e_\varphi = \sum_{nm} \left[ \frac{1}{2} \left( V' + \frac{U - V}{r} \right) \frac{\partial S_{nm}^{(1)}}{\partial \varphi} + \frac{W}{r} \left( \frac{\partial S_{nm}^{(0)}}{\partial \varphi} + N Y_{nm} e_\varphi \right) \right],
\]
and the spherical harmonic expansion of \( \nabla \cdot e \), invoking relations (B.16), (B.40) and (B.42),
\[
\nabla \cdot e = \frac{1}{2} \left[ \nabla \cdot \nabla \mathbf{u} + \nabla \nabla \cdot \mathbf{u} \right] = \frac{1}{2} \left( 2V' r + \frac{4U' - NV'}{r} - \frac{(N + 4)(U - 3NV)}{r^2} \right) S_{nm}^{(1)} + \frac{1}{2} \left( W' r + \frac{2V' r}{r^2} - \frac{NW}{r^2} \right) S_{nm}^{(0)}.
\]
B.5 Elastic Stress Tensor and Elastic Stress Vectors

The elastic stress tensor \( \tau^E \) is defined by the elastic constitutive relation, cf. (2.10),

\[
\tau^E = \lambda \nabla \cdot u + 2\mu e = \lambda \nabla \cdot u + \mu \left[ \nabla u + (\nabla u)^T \right].
\] (B.57)

Tensor \( \tau^E \) is symmetric. To obtain the components of \( \tau^E \), both in the spherical coordinates and in the spherical harmonic expansions, expressions (B.50)–(B.51) for the strain tensor \( e \) can be used,

\[
\begin{bmatrix}
\tau_{rr}^E & \tau_{r\theta}^E & \tau_{r\phi}^E \\
\tau_{\theta r}^E & \tau_{\theta \theta}^E & \tau_{\theta \phi}^E \\
\tau_{\phi r}^E & \tau_{\phi \theta}^E & \tau_{\phi \phi}^E
\end{bmatrix}
= \begin{bmatrix}
\lambda \nabla \cdot u + 2\mu e \\
\lambda \nabla \cdot u + 2\mu e \\
\lambda \nabla \cdot u + 2\mu e
\end{bmatrix}
= \begin{pmatrix}
\lambda X + 2\mu U' \\
\lambda X + 2\mu U' \\
\lambda X + 2\mu U'
\end{pmatrix} Y_{nm} + \mu \begin{pmatrix}
\left(V' + \frac{U-V}{r}\right) Z_{nm} - \left(W' - \frac{W}{r}\right) \tilde{Y}_{nm} \\
\left(V' + \frac{U-V}{r}\right) Z_{nm} - \left(W' - \frac{W}{r}\right) \tilde{Y}_{nm} \\
\left(V' + \frac{U-V}{r}\right) Z_{nm} - \left(W' - \frac{W}{r}\right) \tilde{Y}_{nm}
\end{pmatrix}
\] (B.58)

\[
= \sum_{nm} \begin{pmatrix}
\lambda X + 2\mu U' \\
\lambda X + 2\mu U' \\
\lambda X + 2\mu U'
\end{pmatrix} Y_{nm} + \frac{2\mu}{r} \begin{pmatrix}
V \frac{\partial Z_{nm}}{\partial \theta} + W \left(\frac{\partial Z_{nm}}{\partial \theta} + \frac{2Y_{nm}}{\sin \theta}\right) \\
V \frac{\partial Z_{nm}}{\partial \theta} + W \left(\frac{\partial Z_{nm}}{\partial \theta} + \frac{2Y_{nm}}{\sin \theta}\right) \\
V \frac{\partial Z_{nm}}{\partial \theta} + W \left(\frac{\partial Z_{nm}}{\partial \theta} + \frac{2Y_{nm}}{\sin \theta}\right)
\end{pmatrix}
\] (B.59)

where symbols \( X \) denote the coefficients of the spherical harmonic expansion (B.37) of \( \nabla \cdot u \),

\[
X_{nm} = U'_{nm} + \frac{2U_{nm} - NV_{nm}}{r}. \quad (B.60)
\]

The first invariant of \( \tau^E \) equals to

\[
\tau \equiv \tau_{rr}^E + \tau_{\theta \theta}^E + \tau_{\phi \phi}^E = \sum_{nm} (3\lambda + 2\mu)Y_{nm} = (3\lambda + 2\mu)\nabla \cdot u = 3K \nabla \cdot u. \quad (B.61)
\]

The stress vectors along the directions of \( e_r, e_\theta \) and \( e_\phi \) (with the degree and order subscripts of the stress vector coefficients \( T^E_{r, rnm}, \ldots, T^E_{\phi, rnm} \) suppressed): 

\[
T^E_r \equiv e_r \cdot \tau^E = \tau_{rr}^E e_r + \tau_{r\theta}^E e_\theta + \tau_{r\phi}^E e_\phi = \sum_{nm} \left[ T^E_{rr} S^{(-1)}_{nm} + T^E_{r\theta} S^{(1)}_{nm} + T^E_{r\phi} S^{(0)}_{nm} \right],
\] (B.62)

\[
T^E_\theta \equiv e_\theta \cdot \tau^E = \tau_{\theta r}^E e_r + \tau_{\theta \theta}^E e_\theta + \tau_{\theta \phi}^E e_\phi = \sum_{nm} \left[ T^E_{\theta r} S^{(-1)}_{nm} + T^E_{\theta \theta} S^{(1)}_{nm} + T^E_{\theta \phi} S^{(0)}_{nm} \right],
\] (B.63)

\[
T^E_\phi \equiv e_\phi \cdot \tau^E = \tau_{\phi r}^E e_r + \tau_{\phi \theta}^E e_\theta + \tau_{\phi \phi}^E e_\phi = \sum_{nm} \left[ T^E_{\phi r} S^{(-1)}_{nm} + T^E_{\phi \theta} S^{(1)}_{nm} + T^E_{\phi \phi} S^{(0)}_{nm} \right],
\] (B.64)
with degree and order, it is possible to relate the coefficients \( T \) parameters, \( \lambda \) and \( \rho \) to the coefficients of the spherical harmonic expansions of \( T \). However, no such simple relations are available for the coefficients of the spherical harmonic expansions of \( T \). The expression for \( \nabla \cdot \tau^E \) with the radial distribution of the elastic Lamé parameters, \( \lambda = \lambda(r) \), \( \mu = \mu(r) \), can be found as follows:

\[
\nabla \cdot \tau^E = \lambda \nabla \cdot u + 2 \mu \nabla \cdot e + \lambda' \nabla \cdot u e_r + 2 \mu' e_r \cdot e
\]

\[
= \sum_{nm} \left[ T_{rr}^{E} \left( \frac{4 \mu}{r} U' - \frac{N \mu}{r} V' - \frac{(N + 4) \mu}{r^2} U + \frac{3 N \mu}{r^2} V \right) S_{nm}^{(-1)} + \left( T_{r \theta}^{E} - \frac{\lambda}{r} U' + \frac{3 \mu}{r} V' + \frac{2 \lambda + 5 \mu}{r^2} U - \frac{N \lambda + (2 N + 1) \mu}{r^2} V \right) S_{nm}^{(1)} + \left( T_{r \phi}^{E} + \frac{3 \mu}{r} W' - \frac{(N + 1) \mu}{r^2} W \right) S_{nm}^{(0)} \right]
\]

\[
= \sum_{nm} \left[ T_{rr}^{E} \left( -\frac{4 \mu(3 \lambda + 2 \mu)}{r^2 \beta} U - \frac{2 N \mu(3 \lambda + 2 \mu)}{r^2 \beta} V + \frac{4 \mu}{r^2 \beta} T_{rr}^{E} - \frac{N}{r} T_{r \theta}^{E} \right) S_{nm}^{(-1)} + \left( T_{r \theta}^{E} + \frac{2 \mu(3 \lambda + 2 \mu)}{r^2 \beta} U - \frac{N \mu(3 \lambda + 2 \mu)}{r^2 \beta} V + \frac{\lambda}{r^2 \beta} T_{rr}^{E} + \frac{3 \mu}{r^2 \beta} T_{r \phi}^{E} \right) S_{nm}^{(1)} + \left( T_{r \phi}^{E} - \frac{(N - 2 \mu)}{r^2} W + \frac{3 \mu}{r^2} T_{r \phi}^{E} \right) S_{nm}^{(0)} \right]
\]

with \( \beta = \lambda + 2 \mu \). In (B.69), relations which follows from (B.65)–(B.67),

\[
U' = -\frac{2 \lambda}{r^2 \beta} U + \frac{N \lambda}{r^2 \beta} V + \frac{1}{\beta} T_{rr}^{E},
\]

\[
V' = -\frac{1}{r^2} U + \frac{1}{\mu} V + \frac{1}{\beta} T_{r \theta}^{E},
\]

\[
W' = \frac{1}{r^2} W + \frac{1}{\mu} T_{r \phi}^{E},
\]

\[
X = \frac{2 \mu}{r^2} (2 U - NV) + \frac{1}{\beta} T_{rr}^{E},
\]

have been used.
We rewrite the stress vectors for the zonal functions \( u(r, \vartheta) \), \( \varphi_1(r, \vartheta) \) and \( \tau(r, \vartheta) \), cf. (3.67). The zonal scalar and vector spherical harmonics, respectively, are involved,

\[
Y_n(\vartheta) = Y_{n0}(\vartheta), \quad Z_n(\vartheta) = \frac{\partial Y_n}{\partial \vartheta}, \quad \tilde{Y}(\vartheta) = \frac{1}{\sin \vartheta} \frac{\partial Y_n}{\partial \varphi} = 0, \\
S_n^{(-1)}(\vartheta) = Y_n e_r, \quad S_n^{(1)}(\vartheta) = Z_n e_\vartheta, \quad S_n^{(0)}(\vartheta) = Z_n e_\varphi.
\]

(cf. (B.28)–(B.30)). With relation (B.43) in the reduced form for the zonal spherical harmonics,

\[
\frac{\partial Z_n}{\partial \vartheta} + \frac{\cos \vartheta}{\sin \vartheta} Z_n + NY_n = 0,
\]

the zonal stress vectors can be obtained from (B.62)–(B.64) as follows:

\[
T_r^E = \sum_n \left[ T_{rr,n}^E S_n^{(-1)} + T_{r\vartheta,n}^E S_n^{(1)} + T_{r\varphi,n}^E S_n^{(0)} \right], \\
T_\vartheta^E = \sum_n \left[ T_{\vartheta r,n}^E S_n^{(-1)} + T_{\vartheta\vartheta,n}^E S_n^{(1)} + T_{\vartheta\varphi,n}^E S_n^{(0)} \right], \\
T_\varphi^E = \sum_n \left[ T_{\varphi r,n}^E S_n^{(-1)} + T_{\varphi\vartheta,n}^E S_n^{(1)} + T_{\varphi\varphi,n}^E S_n^{(0)} \right].
\]

### B.6 Other Spherical Harmonic Expansions

In Section 2.2 the spherical harmonic expansions of the forcing term \( f \) in (2.13) and of the l.h.s. of the Poisson equation (2.14) for the incremental field \( \varphi_1 \) are needed. Introducing the spherical harmonic expansion of \( \varphi_1 \),

\[
\varphi_1 = \sum_{nm} F_{nm} Y_{nm},
\]

cf. (2.21), and the auxiliary coefficient \( Q_{nm}(r) \),

\[
Q_{nm} = F_{nm}' + \frac{n+1}{r} F_{nm} + 4\pi G \vartheta_0 U_{nm},
\]

cf. (2.32), the spherical harmonic expansion of \( f \) can be found (the degree and order subscripts of \( F_{nm} \) and \( Q_{nm} \) are suppressed):

\[
f = -\vartheta_0 \nabla \varphi_1 + \nabla \cdot (\vartheta_0 \varphi_1 u) = -\vartheta_0 \nabla \varphi_1 + \vartheta_0 \nabla \cdot (\vartheta_0 \varphi_1 u + \vartheta_0 \nabla (\vartheta_0 \varphi_1 \cdot u)) \\
= \sum_{nm} \left[ -\vartheta_0 F_{nm} S_{nm}^{(-1)} + \frac{\vartheta_0}{r} S_{nm}^{(1)} + \vartheta_0 \vartheta_0 \left( U' + \frac{2U - NV}{r} \right) S_{nm}^{(-1)} - \vartheta_0 \left( (\vartheta_0 U')' S_{nm}^{(-1)} + \frac{\vartheta_0 U}{r} S_{nm}^{(1)} \right) \right] \\
= \sum_{nm} \left[ 4\vartheta_0 \vartheta_0 U_{nm} - N \vartheta_0 \vartheta_0 V_{nm} + \frac{(n+1)\vartheta_0}{r} F_{nm} - \vartheta_0 Q_{nm} \right] S_{nm}^{(-1)} - \left( \frac{\vartheta_0}{r} U_{nm} + \frac{\vartheta_0}{r} F_{nm} \right) S_{nm}^{(1)},
\]
where the Poisson equation (2.4) for the initial field \( \varphi_0 \) in the case of \( \varrho_0 = \varrho_0(r) \) has been used,

\[
g'_0 + \frac{2}{r} g_0 - 4\pi G \varrho_0 = 0 .
\]  

(B.82)

The spherical harmonic expansion of the l.h.s. of the Poisson equation (2.14) can be derived in the two steps:

\[
\nabla \varphi_1 + 4\pi G \varrho_0 \mathbf{u} = \sum_{nm} \left[ \left( F' + 4\pi G \varrho_0 U \right) S_{nm}^{(-1)} + \left( \frac{F}{r} + 4\pi G \varrho_0 V \right) S_{nm}^{(1)} + 4\pi G \varrho_0 W S_{nm}^{(0)} \right],
\]

\[
\nabla \cdot (\nabla \varphi_1 + 4\pi G \varrho_0 \mathbf{u}) = \sum_{nm} \left[ \left( -n + 1 \right) r \varphi + N \right] \frac{2}{r} \left( \frac{n + 1}{r} + Q \right) - \frac{N}{r} \left( \frac{F}{r} + 4\pi G \varrho_0 V \right) Y_{nm},
\]

\[
= \sum_{nm} \left[ Q' + 4\pi G \frac{(n + 1) \varrho_0}{r} U - 4\pi G \frac{N \varrho_0}{r} V - \frac{n - 1}{r} Q \right] Y_{nm}.
\]  

(B.83)

\[\text{f molto espr.}\]

ANTONÍN DVOŘÁK, Concerto for Cello and Orchestra No. 2 in B minor, Op. 104.
II. Adagio, ma non troppo (1895)