



The Method of Lines in Computing Viscoelastic Relaxation of the Earth

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MATHEMATICAL FORMULATION

The Aim

effective evaluation of viscoelastic responses of arbitrarily stratified Earth models to surface loading

Physical Model

Maxwell viscoelastic pre-stressed self-gravitating sphere, compressible or incompressible

Field Partial Differential Equations (PDEs)

momentum equation with Maxwell constitutive relation

$$\partial_t \tau = \mu \nabla \cdot \epsilon + f$$

$$\nabla \cdot (\nabla \varphi + 4\pi G_0 u) = 0$$

with u the displacement, φ the incremental gravitational potential, τ the incremental Cauchy stress tensor, $\epsilon = \nabla u + (\nabla u)^T$, $f = -\rho_0 \nabla \varphi_0 - \rho_1 \nabla \varphi_1 - \nabla \cdot (\rho_0 u \cdot \nabla u)$, $\xi = \mu / \eta$

Peltier, 1974

Normal Mode Approach

(after spherical harmonic decomposition + Laplace transform)

BOUNDARY VALUE PROBLEM IN r parametrized by the Laplace variable s

$$\partial_r y_n(r, s) = A_n(r, s) y_n(r, s)$$

where $y_n(r, s)$ is a 6-vector of spherical harmonic coefficients of u, φ and τ in the Laplace domain, $A_n(r, s)$ is a 6×6 matrix, n is the spherical harmonic degree

TRADITIONAL SOLUTION

- evaluation of a surface secular determinant $\det M_n(s)$, analytically (matrix propagators) or by numerical integration,
- numerical solution (root-finding) of the secular equation $\det M_n(s) = 0$,
- assuming non-degenerate real (and negative) roots in the Laplace domain, exponential relaxation in time is obtained

1999

System of PDEs in Time t and Radius r

(after spherical harmonic decomposition)

$$\partial_t \left[\partial_r y_n(t, r) - A_n y_n(t, r) \right] = \xi \left[D_n \partial_r y_n(t, r) + E_n y_n(t, r) \right]$$

where $y_n(t, r)$ is a 6-vector of spherical harmonic coefficients of u, τ and φ , $A_n(r), D_n(r), E_n(r)$ are 6×6 matrices constant in t , n is the spherical harmonic degree

1999

Initial Value Approach by the Method of Lines (IV/MOL)

SEMI-DISCRETIZATION IN r

INITIAL VALUE PROBLEM IN t

$$P_n \partial_t Y_n(t) = Q_n Y_n(t)$$

a linear system of ordinary differential equations

where $Y_n(t)$ is a $6N$ -vector of $y(t, r_j)$, $j = 1, \dots, N$

P_n, Q_n are constant $[6N]^2$ -matrices

N is a number of grid points

2001

Initial Value Approach by the Method of Rothe (IV/MOR)

SEMI-DISCRETIZATION IN t by Euler's scheme

$$\partial_t \tau \approx \frac{\tau^{i+1} - \tau^i}{\Delta t^i} \text{ or } \partial_t y_n(t, r) \approx \frac{y_n^{i+1}(r) - y_n^i(r)}{\Delta t^i}$$

INITIAL-BOUNDARY VALUE PROBLEM IN r

$$\partial_r y_n^{i+1}(r) - A_n y_n^{i+1}(r) = q_n^i(r)$$

with an explicit expression for $q_n^i(r)$ at the time t^i

1996

1999

1999

2000

IMPLEMENTATION OF THE IV/MOL APPROACH

Discretization in Radius

multidomain Chebyshev staggered grids (see Figures)

high-order pseudospectral schemes for approximation of $y(t, r)$ and $\partial_r y(t, r)$ (Fornberg, 1996)

$$y(t, r, i) = \sum_{j=1}^N \alpha_{ij}^{(n)} y_j(t)$$

$$\partial_r y(t, r, i) = \sum_{j=1}^N \alpha_{ij}^{(n)} \partial_r y_j(t)$$

Stiff Integrators

Rosenbrock and semi-implicit extrapolation methods (Press et al., 1996)

taking into account the block-diagonal structure of P and Q desirable (see next panel)

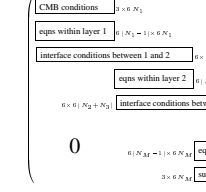
marching in time linear with non-stiff integrators, exponentially increasing with stiff integrators (see Figures)

Block Diagonal Matrices P and Q

For $\partial_t y = f(y)$, stiff integrators require evaluation of $f = hf^{-1}$, $h \sim \Delta t$.

For $P \partial_t Y = QY$, it is enough to evaluate $(P - hQ)^{-1}$, $h \sim \Delta t$,

where both P and Q take the form of a block diagonal matrix



where M is a total number of layers (domains) and N_j is a number of grid points within the j -th layer. A fast LU decomposition of block diagonal matrices can be applied with the CPU time consumption of $O(\sum_{j=1}^M N_j^3)$.

CONCLUSIONS

Highlights of the Method of Lines

- a new formulation of the problem of viscoelastic responses of the Earth to surface loading in the form of the partial differential equation with respect to time and radius,
- offers possibility of testing the stability of a given discretization grid by the matrix eigenvalue analysis (the EV/MOL approach),
- overcomes difficulties of the normal mode approach with the inversion of Laplacian spectra of densely stratified and compressible Earth models,
- overcomes difficulties with restricted time stepping of the initial value approach based on Rothe's method,
- repays by the outstanding speed and accuracy due to applicability of advanced numerical procedures,
- keeps the hope of reasonable handling of laterally varying problems by means of the multidomain Chebyshev decomposition.

References

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MODELS AND RESULTS

Homogeneous Incompressible Sphere

1a) Plot of radial profiles of spherical harmonic coefficients for spherical degree $n = 2$ of u, φ and τ from the elastic start (green) up to a relaxed state in 10^7 yr (blue) with (red) curves showing the relaxation in 100-yr intervals. Discretized in r as a 1-domain (ray) with 20 Chebyshev grid points. Computed within 18 time steps, total CPU time less than 1 sec per spherical degree (Pentium 1 GHz).

b) The same (U_2, V_2), but up to onset of unstable discretization mode after 10^7 yr. CPU time < 1 sec.

c) The eigenspectrum revealed by EV/MOL approach with the stable M0 modal branch (green) and a branch of unstable discretization modes (red). CPU time < 3 sec for 20 grid layers.

Homogeneous Compressible Sphere

2a) The same as in Fig. 1b), but for a compressible sphere and up to only $5 \cdot 10^7$ yr. Reveals the fast onset of an unstable Rayleigh-Taylor mode (growth time ~ 16 kyr).

5-layer Incompressible Earth Model

3a) A model of a benchmark series by Kaufmann et al. (unpublished) called earth-1-a (core, LM, UM and a lithosphere). 4 domains.

b) Time evolution of surface load Love numbers up to 10^8 yr. CPU time for computing this relaxation for 6 selected spherical degrees and 60 grid points amounts to 20 s.

PREM Model with the Anorthosphere

4 A densely stratified PREM based model with $\eta = 10^{22}$ Pa s in LM, 10^{21} Pa s, the 100-km anorthosphere with 10^{19} Pa s and the 120-km elastic lithosphere. With 12 domains and 100 grid points, the CPU requires 10 sec for computing relaxation for first 10^6 yr.

