Cvičení 1 - Vektory a tenzory v E^3

Vektorová ortonormální báze

$\vec{e}_\alpha, \quad \alpha = 1, 2, 3$ vektorová báze

$\vec{e}_\alpha \cdot \vec{e}_\beta = \delta_{\alpha\beta}$ skalární součin

$\vec{e}_\beta \times \vec{e}_\gamma = \epsilon_{\alpha\beta\gamma}\vec{e}_\alpha$ vektorový součin (pravotočivá báze)

Tenzorová ortonormální báze

$\mathbf{e}_{\alpha\beta} = \vec{e}_\alpha \otimes \vec{e}_\beta$ tenzorová báze

Vztah Levi-Civitova symbolu ke Kroneckerovu delta

$$\epsilon_{ijk}\epsilon_{lmn} = \det \begin{pmatrix} \delta_{il} & \delta_{jl} & \delta_{kl} \\ \delta_{im} & \delta_{jm} & \delta_{km} \\ \delta_{in} & \delta_{jn} & \delta_{kn} \end{pmatrix}$$

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

$$\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}$$

$$\epsilon_{ijk}\epsilon_{ijk} = 6$$

$$\mathbf{e}_{\alpha\beta} \cdot \vec{e}_\gamma = (\vec{e}_\alpha \otimes \vec{e}_\beta) \cdot \vec{e}_\gamma = \vec{e}_\alpha(\vec{e}_\beta \cdot \vec{e}_\gamma) = \delta_{\beta\gamma}\vec{e}_\alpha$$

$$\vec{e}_\gamma \cdot \mathbf{e}_{\alpha\beta} = \vec{e}_\gamma \cdot (\vec{e}_\alpha \otimes \vec{e}_\beta) = (\vec{e}_\gamma \cdot \vec{e}_\alpha)\vec{e}_\beta = \delta_{\alpha\gamma}\vec{e}_\beta$$

$$\vec{e}_\alpha \cdot \mathbf{e}_{\beta\gamma} \cdot \vec{e}_\delta = \vec{e}_\alpha \cdot (\vec{e}_\beta \otimes \vec{e}_\gamma) \cdot \vec{e}_\delta = (\vec{e}_\alpha \cdot \vec{e}_\beta)(\vec{e}_\gamma \cdot \vec{e}_\delta) = \delta_{\alpha\beta}\delta_{\gamma\delta}$$

$$\mathbf{e}_{\alpha\beta} \cdot \mathbf{e}_{\gamma\delta} = (\vec{e}_\alpha \otimes \vec{e}_\beta) \cdot (\vec{e}_\gamma \otimes \vec{e}_\delta) = (\vec{e}_\beta \cdot \vec{e}_\gamma)(\vec{e}_\alpha \otimes \vec{e}_\delta) = \delta_{\beta\gamma}(\vec{e}_\alpha \otimes \vec{e}_\delta)$$

$$\mathbf{e}_{\alpha\beta} : \mathbf{e}_{\gamma\delta} = (\vec{e}_\alpha \otimes \vec{e}_\beta) : (\vec{e}_\gamma \otimes \vec{e}_\delta) = (\vec{e}_\beta \cdot \vec{e}_\gamma)(\vec{e}_\alpha \cdot \vec{e}_\delta) = \delta_{\beta\gamma}\delta_{\alpha\delta}$$

Vektory a vektorové operace

… a převod na maticové násobení:

$$\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3 = v_k\vec{e}_k$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = (\text{matice})_{3\times 1} \qquad \vec{v}^T = (v_1\ v_2\ v_3) = (\text{matice})_{1\times 3}$$

Skalární součin

$$\vec{v}\cdot\vec{u} = (v_i\vec{e}_i)\cdot(u_k\vec{e}_k) = v_i u_k(\vec{e}_i\cdot\vec{e}_k) = v_k u_k$$

$$\vec{v}\cdot\vec{u} = \vec{v}^T\vec{u} = (v_1\ v_2\ v_3)\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = (\text{matice } v)_{1\times 3}\ (\text{matice } u)_{3\times 1}$$

$$= \text{matice}_{1\times 1} = \text{skalár} = v_1u_1 + v_2u_2 + v_3u_3$$

Vektorový součin

$$\vec{v}\times\vec{u} = \epsilon_{ijk}\vec{e}_i v_j u_k = (v_2u_3 - v_3u_2)\vec{e}_1$$

$$+(v_3u_1 - v_1u_3)\vec{e}_2 + (v_1u_2 - v_2u_1)\vec{e}_3$$

$$\vec{v}\times\vec{u} = \det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{pmatrix} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Dyadický součin

$$\vec{v}\otimes\vec{u} = (v_i\vec{e}_i)\otimes(u_j\vec{e}_j) = v_i u_j(\vec{e}_i\otimes\vec{e}_j) =$$

$$= v_i u_j \mathbf{e}_{ij} = \begin{pmatrix} v_1u_1 & v_1u_2 & v_1u_3 \\ v_2u_1 & v_2u_2 & v_2u_3 \\ v_3u_1 & v_3u_2 & v_3u_3 \end{pmatrix}$$

$$\vec{v}\otimes\vec{u} = \vec{v}\vec{u}^T = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}(u_1\ u_2\ u_3) = (\text{matice } v)_{3\times 1}\ (\text{matice } u)_{1\times 3}$$

$$= \text{matice}_{3\times 3} = \begin{pmatrix} v_1u_1 & v_1u_2 & v_1u_3 \\ v_2u_1 & v_2u_2 & v_2u_3 \\ v_3u_1 & v_3u_2 & v_3u_3 \end{pmatrix}$$

Tenzory

$$\mathbf{T} = T_{ij}\mathbf{e}_{ij} = T_{ij}(\vec{e}_i \otimes \vec{e}_j) = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

Transponovaný tenzor: $\mathbf{T}^T = T_{ij}(\mathbf{e}_{ij})^T = T_{ij}\mathbf{e}_{ji} = T_{ij}(\vec{e}_j \otimes \vec{e}_i) = T_{ji}(\vec{e}_i \otimes \vec{e}_j) = \begin{pmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{pmatrix}$

Stopa tenzoru (skalár): $\mathrm{tr}\ \mathbf{T} = T_{kk} = T_{11} + T_{22} + T_{33}$

Determinant: $\det\ \mathbf{T} = \epsilon_{ijk} T_{1i} T_{2j} T_{3k} = \frac{1}{3!}\epsilon_{KLM}\epsilon_{klm} T_{kK} T_{lL} T_{mM} = \det \begin{pmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{pmatrix}$

Identická matice: $\mathbf{I} = \delta_{ij}\mathbf{e}_{ij} = \delta_{ij}(\vec{e}_i \otimes \vec{e}_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Inverzní matice: $\mathbf{T}^{-\mathbf{1}} \cdot \mathbf{T} = \mathbf{T} \cdot \mathbf{T}^{-1} = \mathbf{I}\ , \quad (\mathbf{T}^{-\mathbf{1}})_{ij} = \frac{\text{doplněk } T_{ji}}{\det \mathbf{T}} \quad (\text{pouze je}-\text{li } \det \mathbf{T} \neq 0)$

Symetrická matice: $\mathbf{T} = \mathbf{T}^T$

Antisymetrická matice: $\mathbf{T} = -\mathbf{T}^T \quad (\text{je zřejmé, že } \mathrm{tr}\ \mathbf{T} = 0)$

Ortogonální matice: $\mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{Q}^T \cdot \mathbf{Q} = \mathbf{I} \quad (\Rightarrow \mathbf{Q}^T = \mathbf{Q}^{-1} \text{ a } |\det \mathbf{Q}| = 1;\ \det \mathbf{Q} = 1 \rightarrow \text{ rotace},\ \det \mathbf{Q} = -1 \rightarrow \text{ zrcadlení})$

Rozklad tenzoru na symetrickou a antisymetrickou část: $\mathbf{T} = \frac{1}{2}(\mathbf{T} + \mathbf{T}^T) + \frac{1}{2}(\mathbf{T} - \mathbf{T}^T)$

Rozklad tenzoru na izotropní část a deviátor:

$$\mathbf{T} = \tfrac{1}{3}(\mathrm{tr}\ \mathbf{T})\mathbf{I} + [\mathbf{T} - \tfrac{1}{3}(\mathrm{tr}\ \mathbf{T})\mathbf{I}] = \left\{\tfrac{1}{3}T_{kk}\delta_{ij} + [T_{ij} - \tfrac{1}{3}T_{kk}\delta_{ij}]\right\}\mathbf{e}_{ij}$$

$$\mathbf{T} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \begin{pmatrix} T_{11}+p & T_{12} & T_{13} \\ T_{21} & T_{22}+p & T_{23} \\ T_{31} & T_{32} & T_{33}+p \end{pmatrix}\ , \quad \text{kde } p = -\tfrac{1}{3}T_{kk}$$

Rozklad tenzoru na izotropní část, antisymetrickou část a symetrickou bezestopou část:

$$\mathbf{T} = \tfrac{1}{3}(\mathrm{tr}\ \mathbf{T})\mathbf{I} + \tfrac{1}{2}(\mathbf{T} - \mathbf{T}^T) + \tfrac{1}{2}[\mathbf{T} + \mathbf{T}^T - \tfrac{2}{3}(\mathrm{tr}\ \mathbf{T})\mathbf{I}]$$

Násobení tenzorů a vektorů

$$\mathbf{T}\cdot\vec{v} = [T_{ij}(\vec{e}_i\otimes\vec{e}_j)]\cdot(v_k\vec{e}_k) = T_{ij}v_k[(\vec{e}_i\otimes\vec{e}_j)\cdot\vec{e}_k] = T_{ij}v_k\delta_{jk}\vec{e}_i = T_{ij}v_j\vec{e}_i = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}_{3\times3}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{3\times1}$$

$$\vec{v}\cdot\mathbf{T} = (v_k\vec{e}_k)\cdot[T_{ij}(\vec{e}_i\otimes\vec{e}_j)] = T_{ij}v_k[\vec{e}_k\cdot(\vec{e}_i\otimes\vec{e}_j)] = T_{ij}v_k\delta_{ki}\vec{e}_j = T_{ij}v_i\vec{e}_j = T_{ji}v_j\vec{e}_i = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}_{1\times3}\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}_{3\times3}$$

$$\vec{v}\cdot\mathbf{T}\cdot\vec{u} = (v_k\vec{e}_k)\cdot[T_{ij}(\vec{e}_i\otimes\vec{e}_j)]\cdot(v_l\vec{e}_l) = T_{ij}v_ku_l[\vec{e}_k\cdot(\vec{e}_i\otimes\vec{e}_j)\cdot\vec{e}_l] = T_{ij}v_ku_l\delta_{ki}\delta_{jl} = T_{ij}v_iu_j =$$

$$= \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}_{1\times3}\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}_{3\times3}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_{3\times1}$$

$$\mathbf{T}\cdot\mathbf{S} = [T_{ij}(\vec{e}_i\otimes\vec{e}_j)]\cdot[S_{kl}(\vec{e}_k\otimes\vec{e}_l)] = T_{ij}S_{kl}[(\vec{e}_i\otimes\vec{e}_j)\cdot(\vec{e}_k\otimes\vec{e}_l)] = T_{ij}S_{kl}\delta_{jk}(\vec{e}_i\otimes\vec{e}_l) = T_{ij}S_{jl}(\vec{e}_i\otimes\vec{e}_l) =$$

$$= T_{ij}S_{jl}\mathbf{e}_{il} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}_{3\times3}\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}_{3\times3}$$

$$\mathbf{T}:\mathbf{S} = [T_{ij}(\vec{e}_i\otimes\vec{e}_j)]:[S_{kl}(\vec{e}_k\otimes\vec{e}_l)] = T_{ij}S_{kl}[(\vec{e}_i\otimes\vec{e}_j):(\vec{e}_k\otimes\vec{e}_l)] = T_{ij}S_{kl}\delta_{jk}\delta_{il} = T_{ij}S_{ji}$$

Důležité vztahy a tvrzení:

$(\vec{v} \otimes \vec{u}) \cdot \vec{w} = \vec{v}(\vec{u} \cdot \vec{w})$

$\vec{v} \cdot (\vec{u} \otimes \vec{w}) = (\vec{v} \cdot \vec{u})\vec{w}$

$(\vec{v} \otimes \vec{u}) \cdot (\vec{w} \otimes \vec{x}) = (\vec{u} \cdot \vec{w})(\vec{v} \otimes \vec{x})$

$\mathbf{I} \cdot \vec{v} = \vec{v} \cdot \mathbf{I} = \vec{v}$

$\vec{v} \cdot \mathbf{I} \cdot \vec{v} = v^2$

$\mathbf{T} \cdot \vec{v} = \vec{v} \cdot \mathbf{T}^T$

$\mathbf{T} \cdot (\phi\vec{v}) = \phi(\mathbf{T} \cdot \vec{v})$

$\mathbf{T} \cdot \mathbf{I} = \mathbf{I} \cdot \mathbf{T} = \mathbf{T}$

$(\mathbf{T}^{-1})^T = (\mathbf{T}^T)^{-1} =: \mathbf{T}^{-T}$

$\vec{u} \cdot \mathbf{T} \cdot \vec{v} = \mathbf{T} : (\vec{v} \otimes \vec{u})$

$\vec{v} \cdot (\mathbf{A} \cdot \mathbf{B}) = (\vec{v} \cdot \mathbf{A}) \cdot \mathbf{B}$

$\mathbf{A} \cdot (\phi\mathbf{B}) = \phi(\mathbf{A} \cdot \mathbf{B})$

$\mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C}$

$(\mathbf{A} \cdot \mathbf{B}) \cdot (\mathbf{C} \cdot \mathbf{D}) = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) \cdot \mathbf{D}$

$\mathbf{A} : (\mathbf{B} \cdot \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B}) : \mathbf{C} = (\mathbf{C} \cdot \mathbf{A}) : \mathbf{B}$

$(\mathbf{A} \cdot \mathbf{B})^T = \mathbf{B}^T \cdot \mathbf{A}^T$

$(\mathbf{A}^T \cdot \mathbf{B}^T)^T = \mathbf{B} \cdot \mathbf{A}$

$\mathbf{A} \cdot \mathbf{A}^T = (\mathbf{A} \cdot \mathbf{A}^T)^T,$ tj. matice $\mathbf{A} \cdot \mathbf{A}^T$ je symetrická

$\mathbf{A}, \mathbf{B}$ jsou symetrické $\Leftrightarrow$ $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$

$\mathbf{P}, \mathbf{Q}$ jsou ortogonální $\Rightarrow$ $\mathbf{P} \cdot \mathbf{Q}$ je rovněž ortogonální

$(\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1}$

$\mathrm{tr}(\mathbf{A} \cdot \mathbf{B}) = \mathrm{tr}(\mathbf{B} \cdot \mathbf{A})$

$\det \mathbf{I} = 1$

$\det(\mathbf{A} \cdot \mathbf{B}) = \det \mathbf{A} \det \mathbf{B}$

$\det(\phi\mathbf{T}) = \phi^3 \det \mathbf{T}$ pro $\mathbf{T}_{3\times 3}$

$\det(\mathbf{A}^T) = \det \mathbf{A}$

$\det(\mathbf{A}^{-1}) = (\det \mathbf{A})^{-1}$

$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$ Jacobi

$(\vec{a} \times \vec{b}) + (\vec{c} \times \vec{d}) = (\vec{a} - \vec{c}) \times (\vec{b} - \vec{d}) + \vec{a} \times \vec{d} + \vec{c} \times \vec{b}$

$(\mathbf{T} \cdot \vec{a}) \times (\mathbf{T} \cdot \vec{b}) = (\det \mathbf{T})\mathbf{T}^{-T} \cdot (\vec{a} \times \vec{b})$

$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$

$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$

$(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c}) = [\vec{a} \cdot (\vec{b} \times \vec{c})]\vec{a}$

$|\vec{a} \times \vec{b}|^2 = a^2 b^2 - (\vec{a} \cdot \vec{b})^2$ Lagrange

$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$ Binet-Cauchy

Vlastní čísla a vektory matice:

$\mathbf{A} \cdot \vec{p} = \lambda \vec{p}$ ... λ , $\vec{p}$ označujeme jako vlastní čísla resp. vektory matice $\mathbf{A}$

$$\mathbf{A} \cdot \vec{p} - \lambda \vec{p} = (\mathbf{A} - \lambda \mathbf{I}) \cdot \vec{p} = 0$$

charakteristická matice

charakteristická rovnice

netriviální řešení $\Leftrightarrow$ $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$

$-\lambda^3 + E_I \lambda^2 + E_{II} \lambda + E_{III} = 0$ ... charakteristický polynom

$E_I,\ E_{II},\ E_{III}$ ... invarianty matice $\mathbf{A}$

Důležitá tvrzení:

1. Podobné matice mají stejný charakteristický polynom, a tedy stejná vlastní čísla.
2. Je-li $\mathbf{A}$ symetrická a reálná, pak její vlastní čísla jsou reálná. Je-li $\mathbf{A}$ navíc pozitivně definitní, pak všechna vlastní čísla jsou kladná.
3. Nechť $\mathbf{A}$ je symetrická a reálná. Pak existuje taková ortogonální matice $\mathbf{Q}$, že matice $\mathbf{Q}^T \cdot \mathbf{A} \cdot \mathbf{Q}$ je diagonální a reálná. $\mathbf{Q}^T \cdot \mathbf{A} \cdot \mathbf{Q}$ je Jordanův kanonický tvar matice $\mathbf{A}$:

$\mathbf{J} = \mathbf{Q}^T \cdot \mathbf{A} \cdot \mathbf{Q}$ … diagonální matice tvořená vlastními čísly

$\mathbf{Q} = (\vec{p}_1 \mid \vec{p}_2 \mid \vec{p}_3)$ … ortogonální matice tvořená vlastními vektory

Matici $\mathbf{A}$, její determinant a odmocninu (pokud je $\mathbf{A}$ pozitivně definitní) lze pak vyjádřit jako:

$$\mathbf{A} = \mathbf{Q} \cdot \mathbf{J} \cdot \mathbf{Q}^T$$

$$\det \mathbf{A} = \det \mathbf{J}$$

$$\sqrt{\mathbf{A}} = \mathbf{Q} \cdot \sqrt{\mathbf{J}} \cdot \mathbf{Q}^T$$

Diferenciální operátor nabla:

$$\nabla = \vec{e}_k \frac{\partial}{\partial x_k}$$

$$\text{grad } a \equiv \nabla a = \vec{e}_k \frac{\partial a}{\partial x_k} = a_{,k} \vec{e}_k$$

$$\text{grad } \vec{v} \equiv \nabla \otimes \vec{v} = \vec{e}_k \frac{\partial}{\partial x_k} \otimes (v_l \vec{e}_l) = \frac{\partial v_l}{\partial x_k} (\vec{e}_k \otimes \vec{e}_l) = \frac{\partial v_l}{\partial x_k} \mathbf{e}_{kl} = v_{l,k} \mathbf{e}_{kl} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} & \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_1}{\partial x_3} & \frac{\partial v_2}{\partial x_3} & \frac{\partial v_3}{\partial x_3} \end{pmatrix}$$

$$(\text{grad } \vec{v})^T = \frac{\partial v_l}{\partial x_k} (\vec{e}_k \otimes \vec{e}_l)^T = \frac{\partial v_l}{\partial x_k} (\vec{e}_l \otimes \vec{e}_k) = \frac{\partial v_l}{\partial x_k} \mathbf{e}_{lk} = \frac{\partial v_k}{\partial x_l} \mathbf{e}_{kl} = v_{k,l} \mathbf{e}_{kl} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{pmatrix}$$

$$\text{div } \vec{v} \equiv \nabla \cdot \vec{v} = \vec{e}_k \frac{\partial}{\partial x_k} \cdot (v_l \vec{e}_l) = \frac{\partial v_l}{\partial x_k} (\vec{e}_k \cdot \vec{e}_l) = \frac{\partial v_l}{\partial x_k} \delta_{kl} = \frac{\partial v_k}{\partial x_k} = v_{k,k}$$

$$\text{div } \mathbf{T} \equiv \nabla \cdot \mathbf{T} = \vec{e}_k \frac{\partial}{\partial x_k} \cdot [T_{lm}(e_l \otimes \vec{e}_m)] = \frac{\partial T_{lm}}{\partial x_k} [\vec{e}_k \cdot (\vec{e}_l \otimes \vec{e}_m)] = \frac{\partial T_{lm}}{\partial x_k} \delta_{kl} \vec{e}_m = \frac{\partial T_{lm}}{\partial x_l} \vec{e}_m = T_{lm,l} \vec{e}_m$$

$$= \left(\tfrac{\partial}{\partial x_1} \ \tfrac{\partial}{\partial x_2} \ \tfrac{\partial}{\partial x_3} \right)_{1\times 3} \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}_{3\times 3}$$

Diferenciální operátor nabla – užitečné identity:

$$\nabla(ab) = a\nabla b + b\nabla a$$

$$\nabla\cdot(a\vec{v}) = a\nabla\cdot\vec{v} + \vec{v}\cdot\nabla a$$

$$\nabla\times(a\vec{v}) = a\nabla\times\vec{v} + \nabla a\times\vec{v}$$

$$\nabla(\vec{u}\cdot\vec{v}) = \nabla\vec{u}\cdot\vec{v} + \nabla\vec{v}\cdot\vec{u}$$

$$\nabla\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot\nabla\times\vec{u} - \vec{u}\cdot\nabla\times\vec{v}$$

$$\nabla\times(\vec{u}\times\vec{v}) = \vec{v}\cdot\nabla\vec{u} - \vec{u}\cdot\nabla\vec{v} + \vec{u}\ \nabla\cdot\vec{v} - \vec{v}\ \nabla\cdot\vec{u}$$

$$\vec{u}\times(\nabla\times\vec{v}) = \nabla\vec{v}\cdot\vec{u} - \vec{u}\cdot\nabla\vec{v}$$

$$\nabla\times(\nabla a) = 0$$

$$\nabla\cdot(\nabla\times\vec{v}) = 0$$

$$\nabla(\nabla\cdot\vec{v}) = \nabla\cdot[(\nabla\vec{v})^T]$$

$$\nabla\cdot(\nabla\vec{v}) = \nabla(\nabla\cdot\vec{v}) - \nabla\times(\nabla\times\vec{v})$$

$$\nabla(a\vec{v}) = a\nabla\vec{v} + \nabla a\otimes\vec{v}$$

$$\nabla\cdot(a\mathbf{T}) = a\nabla\cdot\mathbf{T} + \nabla a\cdot\mathbf{T}$$

$$\nabla\cdot(\vec{u}\otimes\vec{v}) = \vec{v}\ \nabla\cdot\vec{u} + \vec{u}\cdot\nabla\vec{v}$$

K odvození potřebujeme vztahy pro operace s vektory a tenzory a pravidla pro derivování funkcí.

Příklad:

$$\nabla\cdot(a\mathbf{T}) = (\vec{e}_k\frac{\partial}{\partial x_k})\cdot[a\,T_{lm}(\vec{e}_l\otimes\vec{e}_m)] = [\frac{\partial}{\partial x_k}(a\,T_{lm})]\,[\vec{e}_k\cdot(\vec{e}_l\otimes\vec{e}_m)] = [T_{lm}\frac{\partial a}{\partial x_k}+a\frac{\partial T_{lm}}{\partial x_k}]\,[\vec{e}_k\cdot(\vec{e}_l\otimes\vec{e}_m)] = [T_{lm}\frac{\partial a}{\partial x_k}+a\frac{\partial T_{lm}}{\partial x_k}]\delta_{kl}\vec{e}_m =$$

∇ · $a\mathbf{T}$ · oddělíme derivování a vektorové operace · derivace součinu · provedeme vektorové operace

$$= [T_{lm}\frac{\partial a}{\partial x_l} + a\frac{\partial T_{lm}}{\partial x_l}]\vec{e}_m = \frac{\partial a}{\partial x_l}T_{lm}\vec{e}_m + a\frac{\partial T_{lm}}{\partial x_l}\vec{e}_m = \nabla a\cdot\mathbf{T} + a\nabla\cdot\mathbf{T}$$

$\nabla a\cdot\mathbf{T}$ · $a\nabla\cdot\mathbf{T}$