Role of viscoelasticity in mantle convection models

Vojtěch Patočka¹,², Ondřej Čadek¹, Paul Tackley²
(patocka.vojtech@gmail.com)
¹ Charles University in Prague, ² ETH Zürich

1D Analog – Linear viscoelasticity

- Maxwell element - spring and dashpot in series
- Forces equal: \( F = F_D = F_A \). Deformation adds up: \( \Delta = \Delta_A + \Delta_D \)
- Constitutive relation: \( \frac{\dot{\varepsilon}}{\varepsilon} = 2D \), \( D \) = Strain rate
- Physical equations must be written in terms of geometrical objects - i.e. strains and co/counter-variant tensors of first or higher order.
- Objectivity - material time derivative of a tensor is geometrically not a tensor! (\( A \neq \text{Q}A^T \))
- Only special (objective) stress rate may thus be chosen in the constitutive law

3D Analog – Stress rate obtained as a result

Good dynamic tests often use \( \chi = \frac{\Delta t}{\Delta x} \) for dynamic viscosity - is incorrect. There are four possible strain tensor evolutions/imaginations, fully/limitedly:

\[
\begin{align*}
\sigma^{\alpha} & = \frac{1}{\Delta t} \int_0^{\Delta t} (\sigma(t) - \Delta x - \Delta t) dt \\
\sigma^{\beta} & = \frac{1}{\Delta t} \int_0^{\Delta t} (\sigma(t) + \Delta x + \Delta t) dt \\
\sigma^{\gamma} & = \frac{1}{\Delta t} \int_0^{\Delta t} (\sigma(t) - \Delta x + \Delta t) dt \\
\sigma^{\delta} & = \frac{1}{\Delta t} \int_0^{\Delta t} (\sigma(t) + \Delta x - \Delta t) dt
\end{align*}
\]

which have the following material time derivatives (additive one is equal to the strain rate - which is what originally figures in the definition of viscous deformations):

\[
\begin{align*}
\dot{\varepsilon}^{\alpha} & = 2D - (\sigma^{\beta} + \sigma^{\gamma} - 2\sigma^{\delta}) \\
\dot{\varepsilon}^{\beta} & = 2D - (\sigma^{\alpha} - \sigma^{\gamma} + \sigma^{\delta}) \\
\dot{\varepsilon}^{\gamma} & = 2D - (\sigma^{\alpha} + \sigma^{\beta} - \sigma^{\delta}) \\
\dot{\varepsilon}^{\delta} & = 2D - (\sigma^{\alpha} - \sigma^{\beta} + \sigma^{\gamma})
\end{align*}
\]

Maxwell rheology is based on the following idea: Elastic and viscous deformations add together, but stress is the same for both elastic and viscous parts of the deformation:

\[
\sigma = \sigma^{\text{el}} + \sigma^{\text{vis}}
\]

The choice of \( \chi = \frac{\Delta t}{\Delta x} \) or \( \chi^2 = \frac{\Delta t^2}{\Delta x^2} \) given the relations \( \sigma^{\text{el}} - \sigma^{\text{vis}} \) and thus specifies the rate used in our rheology (only the \( \chi \) we get an objective rate - linear connected rate). In this way we calculate the constrained or constrained (unconstrained) velocity field.

We implement maxwellian viscoelastic rheology into StagYY in order to study how changes in the stress distribution in the lithosphere influence tectonic processes on terrestrial planets. To do so, one needs to tackle a general problem: viscoelastic material responds with instantaneous deformation when forces applied to the material undergo instantaneous change. This results in a critical Deborah number above which simulations experience singularity in the velocity field.

We observe object instability in mantle convection models

Algorithm - time step decoupling

\[
\begin{align*}
\Delta t^{n+1} & = \text{stability}(\Delta t^n, D, \dot{\varepsilon}^{n}, \sigma^n, \sigma^n) \\
\Delta t^{n+1} & = \text{diff}(\Delta t^n - \dot{\varepsilon}^{n-1}, \sigma^{n-1}, \sigma^{n-1}) \\
\dot{\varepsilon}^{n} & = \nabla \left( \frac{\sigma^{n} - \nabla \varepsilon^{n}}{\Delta t} \right) + RaT^n \dot{\varepsilon}_d \\
\sigma^n & = \sigma^{n-1} + \nabla \varepsilon^{n-1}
\end{align*}
\]

Numerical instability – effectively instantaneous deformation

Onset of convection is related to “numerically instantaneous” elastic behavior for Deborah number above certain critical value. This causes a problem since velocities in that part of the domain go to infinity and thus CFL condition stops the simulation. One can set a minimum value for the time step used in the discretized constitutive equation to damp this instability. Locally large velocities then influence only time step used in the thermal equation.

Implementation of visco-elasto-plasticity (with variable shear modulus) into 3D spherical StagYY will follow. We look for suitable benchmarks to test the implementation of elasticity when doing so. Elastic events have to be parametrized to avoid obtaining results with velocities which are outside the scope of infinite Prandtl number convection.

Future directions:

Results – elastic episodes:

Results – stress in stagnant lid:

Viscoelastic vs. viscous deformation in the upper part of the domain

Role of viscoelasticity in mantle convection models