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Wavelet spectra and chaos in thermal convection modelling

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Abstract. Thermal convection in planetary bodies represents a chaotic dynamical system, which can be mathematically described by a set of nonlinear equations. We have studied classical Boussinesq approximation of thermal convection with no inertial forces. Even for a low Rayleigh number (10^5) the convection is chaotic and the wavelet analysis of temperature, kinetic energy and surface heat flow time series demonstrates that wavelet spectra are able to yield an insight into the multiscale temporal dynamics of the convecting systems.

Introduction

Earth's mantle dynamics induced by thermal convection is chaotic on geological time-scales [*Turcotte*, 1992; *Yuen* 1992] as well as on short time-scales [*Li and Nyland*, 1994]. This is the reason why a comparison of thermal anomalies obtained from convection models with, e.g., seismic tomography models, is questionable since seismic tomography yields only a present "snapshot" of the state of the Earth. If we do not have a tool for quantifying the chaos, it is hard to estimate when the chaoticity is so strong that such a comparison does not make sense. This is the reason why quantitative measures of chaoticity could be substantial for unifying convection modelling with other fields of geodynamics. For example, one of such measures is the fractal dimension of the attractor of thermal convection models, which was estimated, e.g., by [*Vincent and Yuen*, 1989] for 2-D convection in Cartesian geometry.

We will concentrate here to the other problem in thermal convection chaoticity: how to quantify a time-evolution of convection fluctuations and their characteristic time-scales. To reveal an internal dynamics of convection systems, we apply the modern technique of the wavelet transform to the time-evolution of both the local quantities (temperature in several selected points) and the global quantities (total heat flow throughout the surface or kinetic energy). We will demonstrate that the wavelet spectra are able to provide detailed information about the time-scales of the convection dynamics.

Description of the convection model

To obtain input time series for the wavelet analysis, we have adopted a 2-D spherical model with the axisymmetry with respect to the z-axis of the coordinate system. The equations describing the studied nonlinear dynamical system are those expressing momentum and energy conservation for incompressible Boussinesq fluid heated purely from below with neglected inertial forces:

$$\nabla \cdot \mathbf{v} = 0 \; , \tag{1}$$

$$\nabla^2 \mathbf{v} - \nabla p + Ra\, T \mathbf{e}_r = 0 \; , \tag{2}$$

$$\frac{\partial T}{\partial t} = \nabla^2 T - \mathbf{v} \cdot \nabla T \; , \tag{3}$$

where $\mathbf{v}(\mathrm{r}, \theta)$ and $T(r, \theta)$ are, respectively, dimensionless velocity and temperature, $p(r, \theta)$ is the dimensionless pressure (these fields are dependent on the radial distance r and the colatitude θ), $\mathbf{e}_r$ is the unit radial vector pointing upward and t is the dimensionless time. We have used impermeable and free-slip boundary conditions. Convection is driven by the temperature difference between the bottom ($T = 1$) and the surface ($T = 0$) of the spherical shell.

The Rayleigh number $Ra = \rho_s \alpha \Delta T g d^3 / \eta_s \kappa_s$, ($\rho_s$ is a reference density, α is the thermal expansivity, ΔT is the temperature drop over the layer, g is the gravity acceleration, d is the layer thickness, η_s represents a reference dynamic viscosity and κ_s is a reference thermal diffusivity) was fixed to the value $Ra = 10^5$ here. The system (1)–(3), where the only nonlinearity is the advection of heat, is the simplest approximation of the thermal convection problems with Ra being the only controlling parameter.

Although Ra corresponding to the whole mantle convection can be about one or two orders higher [*Jarvis and Peltier*, 1989], we will try to demonstrate that the proposed mathematical tools are able to reveal the presence of chaos even in such low Rayleigh number convection. Note here that the situation in the real Earth is even more complex because of the existence of other nonlinearities as the temperature and pressure dependence of viscosity [*Karato*, 1993] as well as of the thermal conductivity [*Hofmeister*, 1999], the stress dependence of viscosity [*van den Berg et al.*, 1993], the radiation of heat [*Matyska et al.*, 1994], dissipation and the adiabatic heating/cooling [e.g. *Hansen et al.*, 1993; *Velímský and Matyska*, 2000]. Moreover, convection patterns can be also influenced by the presence of phase changes.

Details concerning the numerical approach to the system (1)-(3) can be found in the paper by [*Larsen et al.*, 1997]. A stream-function formulation was employed, the spatial derivatives were performed by a 4th order difference schemes [*Fornberg*, 1996], the linear system corresponding to the discretized version of the system (1) and (2) was solved by the ADI method and the time-stepping in (3) has been carried out by a fourth-order explicit Runge-Kutta scheme. We used 100 grid points in the radial direction and 300 grid points in the tangential direction. To test whether the resolution is high enough, we also repeated our computations employing 200 × 600 grid points. We found that the results are robust and the studied chaos is not a result of numerical instabilities.

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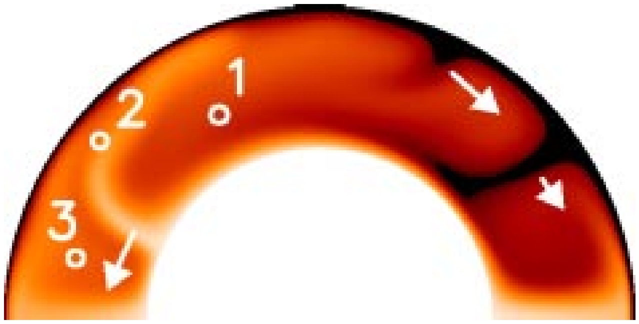

Figure 1. A snapshot of 2-D temperature field at $t = 0.6$ with the three points chosen for the analysis of temperature time series. The axis of axisymmetry is horizontal. The short arrow indicates very slow shift of the main downwelling whereas the long arrows show the direction of fast motions of repeatedly generated local upwellings and downwellings.

Wavelet analysis

We have chosen the wavelet analysis based on the Morlet mother wavelet [e.g. *Torrence and Compo*, 1998], which is defined by

$$\Psi_o(t) = \pi^{-\frac{1}{4}} \exp(i\omega_o t) \exp(-\frac{t^2}{2}) \ , \qquad (4)$$

where the parameter ω_o was fixed to 6. The continuous wavelet transform of a signal $x(t)$ is defined by

$$(T^w x)(a,b) = \int_{-\infty}^{+\infty} x(t) a^{-\frac{1}{2}} \Psi_o^*(\frac{t-b}{a})\, dt \ , \qquad (5)$$

where a represents scaling (characteristic time), b is the time-shift and the star denotes the complex conjugation. This kind of analysis enables to employ efficiently the Fast Fourier Transform in evaluating (5) and study thus the broad range of time-scales limited only by the time-stepping in numerical integration of (3) on the lower side and by the length of time-integration on the upper side.

As there is a number of different wavelets which could be, in principle, employed, the question arises what are the reasons for this particular choice. This is discussed in detail by [e.g. *Torrence and Compo*, 1998], who showed that this choice of complex wavelet transform is useful for time series analysis, where smooth, continuous variations in wavelet amplitude spectrum are expected due to oscillatory behavior of studied time series.

To make it easier to compare the wavelet spectra with the Fourier spectra, we will replace the scaling parameter a by the Fourier period τ. The relation between τ and a is [e.g. *Torrence and Compo*, 1998]

$$\tau = \frac{4\pi a}{\omega_o + \sqrt{2 + \omega_o^2}} \ . \qquad (6)$$

First, we analysed time series of temperature in several different points. We have chosen here the three points, see Fig. 1, to demonstrate that temperature variations can be strongly position-dependent. The axis of axisymmetry in Fig. 1 is horizontal and the convection is dominated by two opposite major plumes lying below the polar regions and one cold downwelling, which is visible in the right part of the figure. In principle, there were two convection cells. The aspect ratio of the left cell was about four, which led to the periodic generation of instabilities in the lower and upper boundary layers. The hot instabilities were then attracted to the major plume and the cold instabilities ended their lives by conjunction with the major downwelling. Point 1 was located in the middle of the cell.

The wavelet analysis of the temperature fluctuations in point 1 (see Fig. 2) reveals that the dominant periods are higher than 0.1, which is not directly visible from the original time series. This time-scale corresponds to very slow changes in the convection like those associated with the motion of the major downwelling. This long-time feature is also clearly visible in the Fourier spectrum, however, the wavelet spectrum shows more: the splitting of long time-scales during the time interval $\langle 0.5, 0.65 \rangle$ and changes of short-time oscillations during the evolution of the studied system. Point 2 is closer to the major plume and thus it is more influenced by the bottom boundary layer instabilities. Although the Fourier spectrum of the temperature fluctuations in this point (see Fig. 3) suggests that slow processes are only of the minor importance, the wavelet spectrum proved that a narrow band of long time-scales is still important in this point. Point 3 is very close to the plume and thus the temperature fluctuations in this point are strongly influenced by plume-plume interactions, which result in very complicated pattern of the spectra, see Fig. 4.

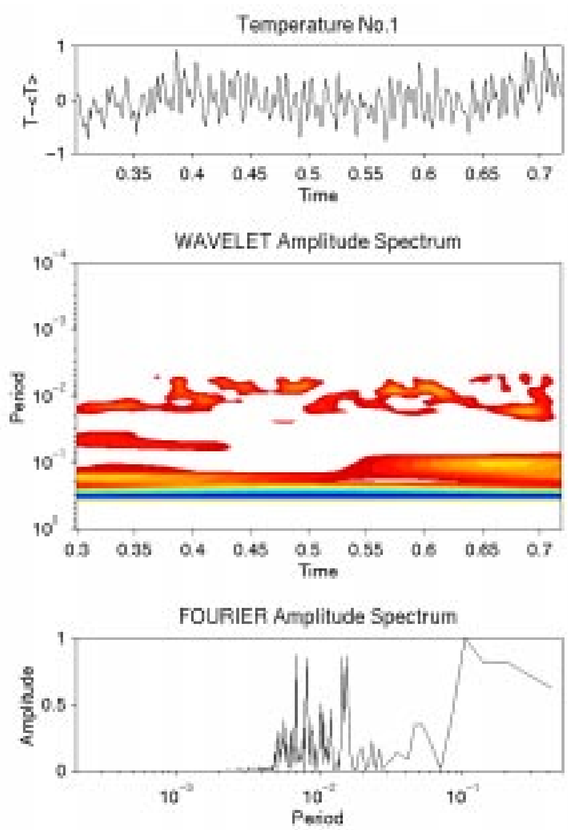

Figure 2. Temperature time series $T - \langle T \rangle$ (normalized by the maximum) in the point 1, where $\langle T \rangle$ denotes the average temperature, its amplitude wavelet spectrum and amplitude Fourier spectrum (from top to bottom). The time series for $t < 0.3$ was excluded from the analysis to eliminate the influence of a transition state.

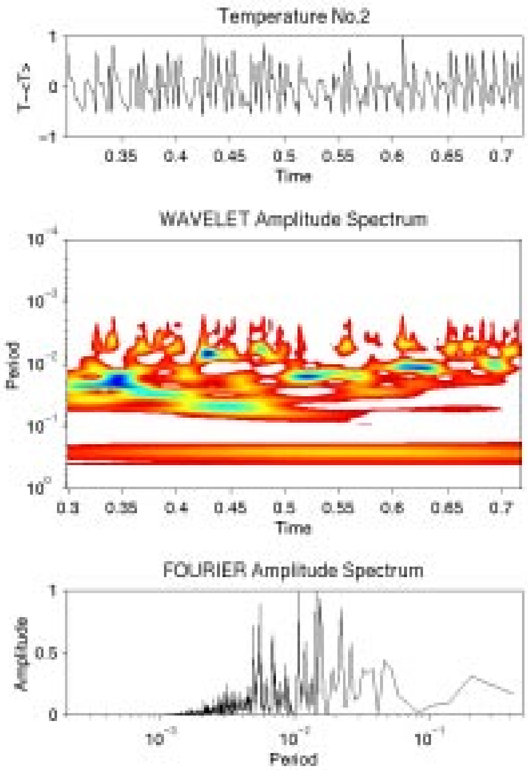

Figure 3. Same as in Fig. 2 but for the point 2.

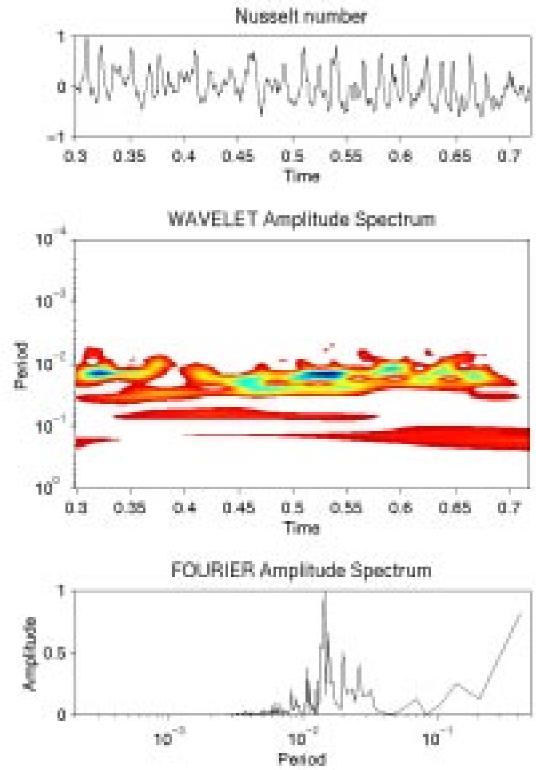

Figure 5. Total surface heat flow fluctuations (zero corresponds to the average value of heat flow), its amplitude wavelet spectrum and amplitude Fourier spectrum (from top to bottom). Otherwise, same as in Fig. 2.

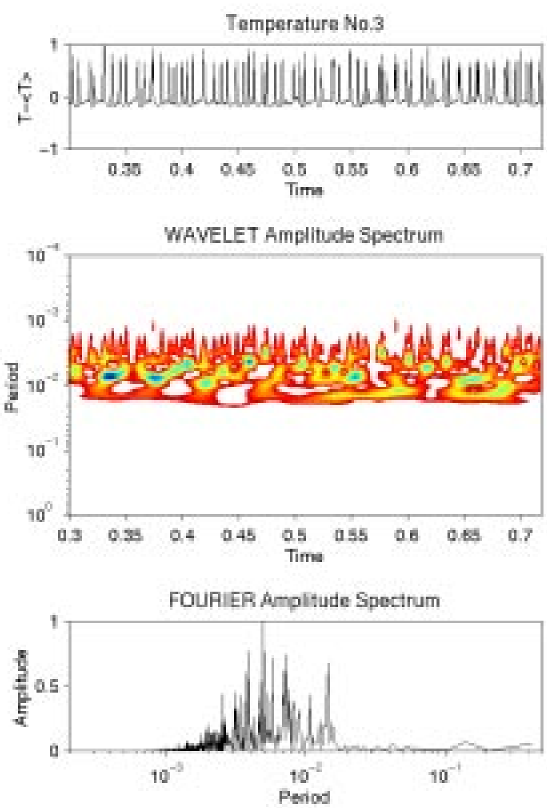

Figure 4. Same as in Fig. 2 but for the point 3.

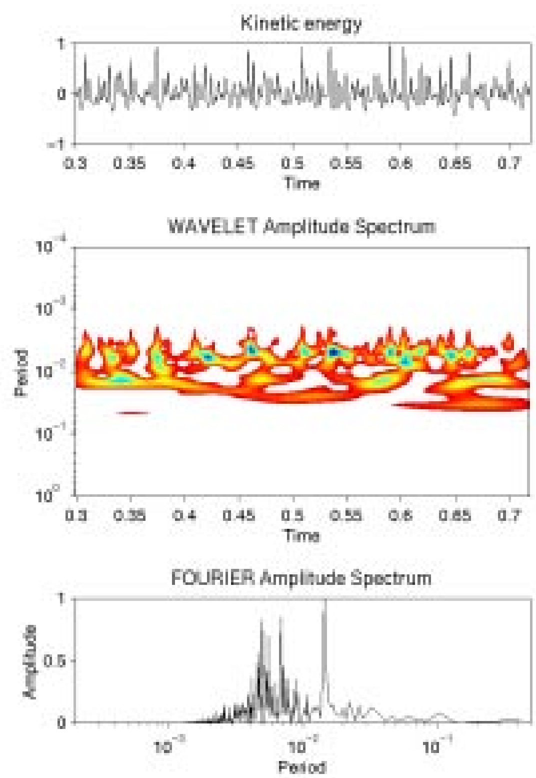

Figure 6. Kinetic energy fluctuations (zero corresponds to its average value), its amplitude wavelet spectrum and amplitude Fourier spectrum (from top to bottom). Otherwise, same as in Fig. 2.

Secondly, we analysed fluctuations of the two integral quantities: the total surface heat flow and the total kinetic energy. Both the wavelet spectrum and the Fourier spectrum of the surface heat flow fluctuations in Fig. 5 show an interesting feature: the shortest time-scales with periods lower than 0.005 are not significant. This probably corresponds to the diffusion character of heat transfer at the surface, when damping of the surface heat fluctuations generated by oscillatory behavior inside the convecting layer increases with decreasing characteristic times of the processes. On the other hand, one can see from Fig. 6 that the kinetic energy of the flow is very sensitive to the behavior of the boundary layer instabilities with short characteristic time-scales because the spectra of the kinetic energy are very similar to the temperature spectra in Fig. 4. Emphasize that prevailing characteristic times in these two figures are strongly fluctuating quantities throughout the evolution of the system, which points to a complicated dynamics of the boundary layers. Such features could not be revealed by means of the traditional Fourier analysis, which is frequently used for analysis of data generated by nonlinear dynamical systems.

Concluding remarks

Although the Rayleigh number was chosen relatively low, the system exhibits chaotic behavior. This result suggests that real mantle convection can be characterized by a high degree of chaoticity on geological time-scales. The characteristic time-scales of non-stationary processes extend over two orders of magnitude. It is remarkable that fast pulsations are clearly detectable in the kinetic energy, where the influence of slow processes is negligible. On the other hand, slow processes are still detectable in the surface heat flow. Time behavior of temperature is strongly position-dependent: there are locations where the temperature evolution is dominated by slow processes but in other places fast pulsations prevail. There is a distinct time-scale gap between these two processes, which points to multivariability of internal planetary dynamics.

Recently Bergeron et al. [1999] demonstrated that Gaussian wavelets are capable to extract tectonic objects from seismic tomography models, i.e, the wavelets are suitable for the analysis of spatial objects in geodynamics. Here we have concentrated on the time-dependence of geodynamic processes and we may conclude that the wavelet analysis is the sensitive tool also for evaluation of their multiscale time-behavior.

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