On models for viscoelastic fluid-like materials that are mechanically incompressible and thermally compressible or expansible and their Oberbeck–Boussinesq type approximations

Vít Průša (joint work with K.R. Rajagopal)
prusv@karlin.mff.cuni.cz

Mathematical Institute, Charles University

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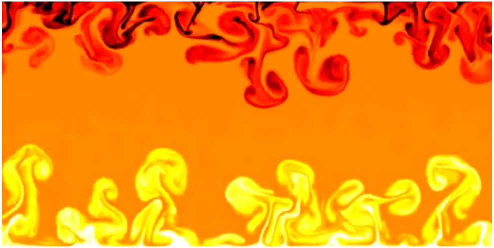
Buoyancy driven flows



Oberbeck–Boussinesq approximation

Original papers:

- J. Boussinesq. *Théorie analytique de la chaleur*. Gauthier-Villars, Paris, 1903
- A. Oberbeck. Über die Wärmeleitung der Flüssigkeiten bei Berücksichtigung der Strömungen infolge von Temperaturdifferenzen. *Ann. Phys. Chem.*, 1:271, 1879

Oberbeck–Boussinesq approximation

Governing equations:

$$
\begin{aligned}
\operatorname{div} \mathbf{v} &= 0 \\
\rho_{\mathrm{ref}} \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= -\nabla m + \mu_{\mathrm{ref}} \Delta \mathbf{v} + \rho_{\mathrm{ref}} (1 + \alpha(\theta - \theta_{\mathrm{ref}}))\mathbf{b} \\
\rho_{\mathrm{ref}} c_{m,\mathrm{ref}} \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \kappa \Delta \theta
\end{aligned}
$$

Rayleigh number:

$$
\mathrm{Ra} =_{\mathrm{def}} \frac{\rho_{\mathrm{ref}} g \alpha_{\mathrm{ref}} \theta_{\mathrm{diff}} l_{\mathrm{char}}^3}{k \eta_{1,\mathrm{ref}}}
$$

Full system of governing equations

Governing equations:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \operatorname{div} \mathbf{v} = 0$$

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \operatorname{div} \mathbb{T} + \rho \mathbf{b}$$

$$\rho \frac{\mathrm{d}e}{\mathrm{d}t} = \mathbb{T} : \mathbb{D} - \operatorname{div} \mathbf{q}$$

Constitutive relations:

$$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D}$$

$$\mathbf{q} = -\kappa \nabla \theta$$

$$p = p(\rho, \theta)$$

$$e = e(\dots)$$

Oberbeck–Boussinesq approximation

- E. A. Spiegel and G. Veronis. On the Boussinesq approximation for a compressible fluid. *Astrophys. J.*, 131:442–447, 1960

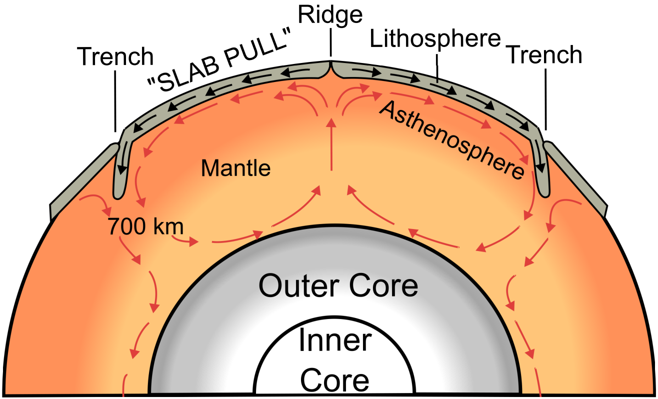
  > *In equation (19) we have retained the term* $g\varepsilon(\rho'/\Delta\rho_0)\mathbf{k}$ *even through it contains* ε *as a factor.*

- John M. Mihaljan. A rigorous exposition of the Boussinesq approximations applicable to a thin layer of fluid. *Astrophys. J.*, 136:1126–1133, 1962
- K. R. Rajagopal, M. Růžička, and A. R. Srinivasa. On the Oberbeck–Boussinesq approximation. *Math. Models Methods Appl. Sci.*, 6(8):1157–1167, 1996

Oberbeck–Boussinesq system as an approximation of an “exact” system

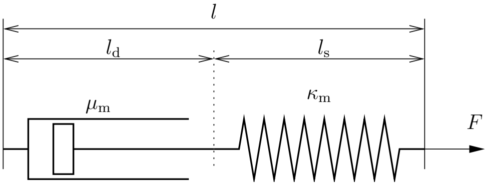
- Mechanically and thermally compressible/expansible material ($\mathrm{Ma} \approx \varepsilon$, $\mathrm{Fr} \approx \sqrt{\varepsilon}$).
  Eduard Feireisl and Antonin Novotný. The Oberbeck–Boussinesq approximation as a singular limit of the full Navier–Stokes–Fourier system. *J. Math. Fluid Mech.*, 11:274–302, 2009
- Mechanically incompressible and thermally compressible/expansible material.
  K. R. Rajagopal, M. Růžička, and A. R. Srinivasa. On the Oberbeck–Boussinesq approximation. *Math. Models Methods Appl. Sci.*, 6(8):1157–1167, 1996

Buoyancy driven flows in viscoelastic fluids



Source: Wikipedia

Viscoelastic fluids – Maxwell model



Mechanical analogue:

- Spring – energy storage.
- Dashpot – energy dissipation.

Problems:

- Distribution of the deformation between the elements (spring, dashpot).
- Three dimensional model. (*Galilean invariance*.)
- Meets laws of thermodynamics. (No *perpetual motion*.)

Upper convected Maxwell model

A three-dimensional incompressible viscoelastic rate type model:

$$\mathbb{T} = -p\mathbb{I} + \mathbb{S}$$
$$\tau \overset{\triangledown}{\mathbb{S}} + \mathbb{S} = 2\mu\mathbb{D}$$
$$\operatorname{div} \mathbf{v} = 0$$

$$\mathbb{D} =_{\mathrm{def}} \frac{1}{2}\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^{\top}\right)$$
$$\mathbb{L} =_{\mathrm{def}} \nabla \mathbf{v}$$
$$\overset{\triangledown}{\mathbb{A}} =_{\mathrm{def}} \frac{\mathrm{d}\mathbb{A}}{\mathrm{d}t} - \mathbb{L}\mathbb{A} - \mathbb{A}\mathbb{L}^{\top}$$

Oberbeck–Boussinesq approximation for viscoelastic fluids

Governing equations:

$$
\begin{aligned}
\operatorname{div} \mathbf{v} &= 0 \\
\rho_{\mathrm{ref}} \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \operatorname{div} \mathbb{T} + \rho_{\mathrm{ref}} (1 + \alpha(\theta - \theta_{\mathrm{ref}}))\mathbf{b} \\
\rho_{\mathrm{ref}} c_{m,\mathrm{ref}} \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \kappa \Delta \theta
\end{aligned}
$$

Naive approach is to replace Navier–Stokes fluid model

$$
\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D}
$$

by Maxwell model

$$
\begin{aligned}
\mathbb{T} &= -p\mathbb{I} + \mathbb{S} \\
\tau \overset{\triangledown}{\mathbb{S}} + \mathbb{S} &= 2\mu\mathbb{D} \\
\operatorname{div} \mathbf{v} &= 0
\end{aligned}
$$

Full model

Requirements:

- Mechanically incompressible but thermally compressible/expansible materials. (Liquids.)
- Different nature of the pressure. (No equation of state. Liquids.)
- Viscoelastic fluid like materials. (Internal energy has a non-thermal contribution.)
- Interplay between all the effects.

Simplification:

- Let us consider only constant material coefficients. (Temperature independent viscosity.)

Plan

Viscoelasticity:

- K. R. Rajagopal and A. R. Srinivasa. A thermodynamic frame work for rate type fluid models. *J. Non-Newton. Fluid Mech.*, 88(3):207–227, 2000

Mechanical incompressibility and thermal expansivity/compressibility:

- K. R. Rajagopal, M. Růžička, and A. R. Srinivasa. On the Oberbeck–Boussinesq approximation. *Math. Models Methods Appl. Sci.*, 6(8):1157–1167, 1996

Example – viscous fluids

Internal energy *ansatz*:

$$e(\rho, \eta) = g(\rho)\eta + h(\rho)$$

Time derivative:

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{\partial g}{\partial \rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}\eta + g\frac{\mathrm{d}\eta}{\mathrm{d}t} + \frac{\partial h}{\partial \rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}$$

Entropy production:

$$\rho\frac{\mathrm{d}\eta}{\mathrm{d}t} + \operatorname{div}\frac{\mathbf{q}}{\theta} = \frac{1}{\theta}\left(-\mathbf{q} \bullet \nabla\theta + \mathbb{T}_\delta : \mathbb{D}_\delta + \left(\alpha\rho^2\frac{\partial h}{\partial \rho} - \rho\eta - \alpha m\right)\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)$$

Second law satisfied if:

$$\mathbf{q} =_{\mathrm{def}} -\kappa\nabla\theta, \qquad \mathbb{T}_\delta =_{\mathrm{def}} 2\mu\mathbb{D}_\delta \qquad \alpha\rho^2\frac{\partial h}{\partial \rho} - \rho\eta - \alpha m =_{\mathrm{def}} 0$$

Viscous fluids – result

$$\Gamma_1 \frac{\mathrm{d}\vartheta^\star}{\mathrm{d}t^\star} = \operatorname{div}^\star \mathbf{v}^\star$$

$$\rho^\star \frac{\mathrm{d}\mathbf{v}^\star}{\mathrm{d}t^\star} = \Gamma_2 \left(\Delta^\star \mathbf{v}^\star + \frac{1}{3} \nabla^\star (\operatorname{div}^\star \mathbf{v}^\star) \right) + \frac{1}{\Gamma_1} \left(-\nabla^\star m^\star + \rho^\star \mathbf{b}^\star \right)$$

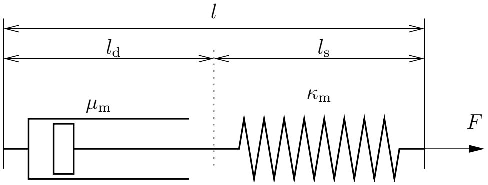
$$\Gamma_4 \Gamma_2 \rho^\star c_m^\star \frac{\mathrm{d}\vartheta^\star}{\mathrm{d}t^\star} = \Gamma_3 \Gamma_2 \Delta^\star \vartheta^\star + 2\Gamma_2 \mathbb{D}_\delta : \mathbb{D}_\delta + \left(\vartheta^\star + \frac{\theta_{\mathrm{ref}}}{\theta_{\mathrm{diff}}} \right) \frac{\mathrm{d}m^\star}{\mathrm{d}t^\star}$$

Parameter values

	Experiment				
	Water	Water	Water	Glycerol	Mercury
Γ_1	2.40×10^{-5}	7.64×10^{-4}	1.55×10^{-2}	1.68×10^{-2}	1.82×10^{-3}
Γ_2	2.17×10^{-1}	3.84×10^{-2}	1.39×10^{-5}	7.48×10^{-3}	2.70×10^{-5}
Γ_3	6.68×10^{7}	6.68×10^{7}	1.01×10^{6}	1.15×10^{3}	3.16×10^{7}
Γ_4	2.19×10^{9}	1.23×10^{10}	3.18×10^{11}	3.91×10^{8}	2.89×10^{10}
Ra	1.5×10^{2}	4.8×10^{3}	2.3×10^{10}	4.5×10^{7}	3.4×10^{7}
Pr	7.1	7.1	4.4	2.5×10^{3}	2.5×10^{-2}
Re	4.6	2.6×10^{1}	7.2×10^{4}	1.3×10^{2}	3.7×10^{4}

Table: Dimensionless parameters in some experiments with viscous fluids.

Viscoelastic fluids – Maxwell model



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Problems:

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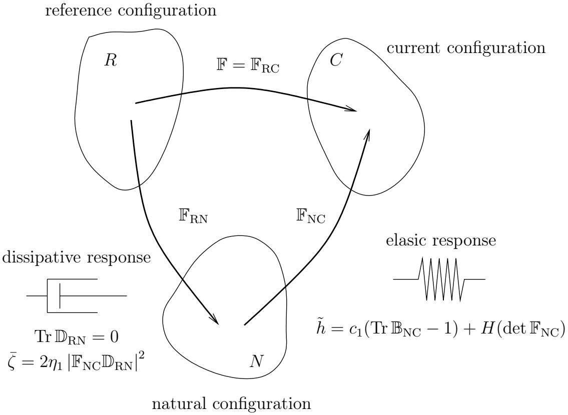
- Meets laws of thermodynamics. (No *perpetual motion*.)

Notion of natural configuration



Problem:

- Distribution of the deformation between the dissipative and elastic response. (Condition $\mathbb{F} = \mathbb{F}_{\mathrm{NC}}\mathbb{F}_{\mathrm{RN}}$ is not enough restrictive.)

Evolution equation for the elastic response

Evolution of the elastic response:

$$\frac{\mathrm{d}\mathbb{B}_{\mathrm{NC}}}{\mathrm{d}t} = \mathbb{L}\mathbb{B}_{\mathrm{NC}} + \mathbb{B}_{\mathrm{NC}}\mathbb{L}^{\top} - 2\mathbb{F}_{\mathrm{NC}}\mathbb{D}_{\mathrm{RN}}\mathbb{F}_{\mathrm{NC}}^{\top}$$

Oldroyd upper convected derivative:

$$\overset{\triangledown}{\mathbb{A}} =_{\mathrm{def}} \frac{\mathrm{d}\mathbb{A}}{\mathrm{d}t} - \mathbb{L}\mathbb{A} - \mathbb{A}\mathbb{L}^{\top}.$$

Evolution equation for the elastic response:

$$\overset{\triangledown}{\mathbb{B}}_{\mathrm{NC}} = -2\mathbb{F}_{\mathrm{NC}}\mathbb{D}_{\mathrm{RN}}\mathbb{F}_{\mathrm{NC}}^{\top}$$

Evolution equation for the internal energy

Internal energy *ansatz*:

$$e = g(\rho)\eta + \overline{h}(\tilde{\mathsf{I}}_1(\mathbb{B}_{\mathrm{NC}}), \tilde{\mathsf{I}}_2(\mathbb{B}_{\mathrm{NC}}), \det \mathbb{F}_{\mathrm{NC}})$$

Evolution equation for the internal energy:

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{\partial g}{\partial \rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}\eta + g\frac{\mathrm{d}\eta}{\mathrm{d}t} + \frac{\partial \overline{h}}{\partial \tilde{\mathsf{I}}_1}\frac{\mathrm{d}\tilde{\mathsf{I}}_1}{\mathrm{d}t} + \frac{\partial \overline{h}}{\partial \tilde{\mathsf{I}}_2}\frac{\mathrm{d}\tilde{\mathsf{I}}_2}{\mathrm{d}t} + \frac{\partial \overline{h}}{\partial \det \mathbb{F}_{\mathrm{NC}}}\frac{\mathrm{d}}{\mathrm{d}t}(\det \mathbb{F}_{\mathrm{NC}})$$

Useful identities:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det \mathbb{F}_{\mathrm{NC}}) = \det \mathbb{F}_{\mathrm{NC}}(\operatorname{Tr} \mathbb{D} - \operatorname{Tr} \mathbb{D}_{\mathrm{RN}})$$

$$\frac{\partial \tilde{h}}{\partial \tilde{\mathsf{I}}_1} \operatorname{Tr} \frac{\mathrm{d}\mathbb{B}_{\mathrm{NC}}}{\mathrm{d}t} = 2\frac{\partial \tilde{h}}{\partial \tilde{\mathsf{I}}_1}\left(\mathbb{B}_{\mathrm{NC}} : \mathbb{D} - \operatorname{Tr}\left(\mathbb{F}_{\mathrm{NC}}\mathbb{D}_{\mathrm{RN}}\mathbb{F}_{\mathrm{NC}}^{\top}\right)\right)$$

Evolution equation for the entropy

Evolution equation for the entropy:

$$
\begin{aligned}
\rho\theta\frac{\mathrm{d}\eta}{\mathrm{d}t} = &-\operatorname{div}\mathbf{q} + \left(\mathbb{T}_\delta - 2\rho\frac{\partial\overline{h}}{\partial\tilde{\mathsf{I}}_1}(\mathbb{B}_{\mathrm{NC}})_\delta\right) : \mathbb{D}_\delta \\
&\qquad + 2\rho\frac{\partial\overline{h}}{\partial\tilde{\mathsf{I}}_1}\operatorname{Tr}\left(\mathbb{F}_{\mathrm{NC}}\mathbb{D}_{\mathrm{RN}}\mathbb{F}_{\mathrm{NC}}^\top\right) \\
&+ \left(-\alpha\rho\frac{\partial\overline{h}}{\partial\det\mathbb{F}_{\mathrm{NC}}}\det\mathbb{F}_{\mathrm{NC}} - \rho\eta - \alpha m - \frac{2}{3}\rho\alpha\frac{\partial\overline{h}}{\partial\tilde{\mathsf{I}}_1}\operatorname{Tr}\mathbb{B}_{\mathrm{NC}}\right)\frac{\mathrm{d}\theta}{\mathrm{d}t} \\
&\qquad + \rho\frac{\partial\overline{h}}{\partial\det\mathbb{F}_{\mathrm{NC}}}(\det\mathbb{F}_{\mathrm{NC}})\operatorname{Tr}\mathbb{D}_{\mathrm{RN}}
\end{aligned}
$$

Refinement of constitutive assumptions

"Solid" part is a compressible neo-Hookean elastic solid:

$$\tilde{h}(\tilde{\mathrm{I}}_1, \tilde{\mathrm{I}}_2, \det \mathbb{F}_{\mathrm{NC}}) =_{\mathrm{def}} c_1 \left(\tilde{\mathrm{I}}_1 - 3\right) + H(\det \mathbb{F}_{\mathrm{NC}})$$

"Fluid" part is incompressible:

$$\operatorname{Tr} \mathbb{D}_{\mathrm{RN}} =_{\mathrm{def}} 0$$

Fourier's law:

$$\mathbf{q} =_{\mathrm{def}} -\kappa \nabla \theta$$

Cauchy stress tensor:

$$\mathbb{T}_{\delta} =_{\mathrm{def}} 2\rho \frac{\partial \tilde{h}}{\partial \tilde{\mathrm{I}}_1} (\mathbb{B}_{\mathrm{NC}})_{\delta}$$

Entropy:

$$\alpha \rho^2 \frac{\partial \tilde{h}}{\partial \rho} - \rho \eta - \alpha m - \frac{2}{3} \rho \alpha \frac{\partial \tilde{h}}{\partial \tilde{\mathrm{I}}_1} \operatorname{Tr} \mathbb{B}_{\mathrm{NC}} =_{\mathrm{def}} 0$$

Entropy production

Entropy production:

$$\varsigma = 2\rho \frac{\partial \tilde{h}}{\partial \tilde{\mathsf{I}}_1} \operatorname{Tr}\left(\mathbb{F}_{\mathrm{NC}} \mathbb{D}_{\mathrm{RN}} \mathbb{F}_{\mathrm{NC}}^{\top}\right) + \kappa \frac{\left|\nabla \theta\right|^2}{\theta}$$

Entropy production

Entropy production:

$$\varsigma = 2\rho \frac{\partial \tilde{h}}{\partial \tilde{\mathsf{I}}_1} \operatorname{Tr}\left(\mathbb{F}_{\mathrm{NC}} \mathbb{D}_{\mathrm{RN}} \mathbb{F}_{\mathrm{NC}}^{\top}\right) + \kappa \frac{\left|\nabla \theta\right|^2}{\theta}$$

Constitutive assumption:

$$\overline{\varsigma} =_{\mathrm{def}} 2\eta_1 \left|\mathbb{F}_{\mathrm{NC}} \mathbb{D}_{\mathrm{RN}}\right|^2$$

Maximization procedure leads to:

$$\mathbb{F}_{\mathrm{NC}} \mathbb{D}_{\mathrm{RN}} \mathbb{F}_{\mathrm{NC}}^{\top} = \frac{\rho}{\eta_1} \frac{\partial \tilde{h}}{\partial \tilde{\mathsf{I}}_1} \left(\mathbb{B}_{\mathrm{NC}} - \overline{\lambda} \mathbb{I}\right)$$

Recall:

$$\overset{\triangledown}{\mathbb{B}}_{\mathrm{NC}} = -2 \mathbb{F}_{\mathrm{NC}} \mathbb{D}_{\mathrm{RN}} \mathbb{F}_{\mathrm{NC}}^{\top}$$

Generalized Maxwell model

Introduce notation:

$$\overline{\mathbb{S}} =_{\mathrm{def}} \rho\mu_1 \left(\mathbb{B}_{\mathrm{NC}} - \overline{\lambda}\mathbb{I}\right) \qquad p =_{\mathrm{def}} m + \rho\mu_1 \left(\frac{1}{3}\operatorname{Tr}\mathbb{B}_{\mathrm{NC}} - \overline{\lambda}\right)$$

Generalized Maxwell model:

$$\mathbb{T} = -p\mathbb{I} + \overline{\mathbb{S}}$$

$$\frac{\eta_1}{\rho\mu_1}\overset{\triangledown}{\overline{\mathbb{S}}} + \left(1 + \frac{\eta_1}{\rho\mu_1}\operatorname{div}\mathbf{v}\right)\overline{\mathbb{S}} = 2\eta_1\overline{\lambda}\mathbb{D}_\delta + \frac{2}{3}\eta_1\overline{\lambda}\left(\operatorname{div}\mathbf{v}\right)\mathbb{I} - \eta_1\frac{\mathrm{d}\overline{\lambda}}{\mathrm{d}t}\mathbb{I}$$

Viscoelastic fluids – result

$$\Lambda_1 \frac{\mathrm{d}\vartheta^\star}{\mathrm{d}t^\star} = \operatorname{div}^\star \mathbf{v}^\star$$

$$\frac{1}{\mathrm{Pr}} \frac{\mathrm{Ra}^{\frac{2}{3}}}{\mathrm{Ra}_{\mathrm{crit}}^{\frac{1}{6}}} \rho^\star \frac{\mathrm{d}\mathbf{v}^\star}{\mathrm{d}t^\star} = \operatorname{div}^\star \overline{\mathbb{S}}^\star{}_\delta + \frac{\mathrm{Ra}^{\frac{1}{3}} \mathrm{Ra}_{\mathrm{crit}}^{\frac{1}{6}}}{\Lambda_1} \left(-\nabla^\star m^\star + \rho^\star \mathbf{b}^\star \right)$$

$$\Lambda_2 \overset{\triangledown\star}{\overline{\mathbb{S}}^\star} + \left(\rho^\star + \Lambda_2 \operatorname{div}^\star \mathbf{v}^\star\right) \overline{\mathbb{S}}^\star = 2\rho^\star \overline{\lambda} \left(\mathbb{D}^\star{}_\delta + \frac{1}{3} \left(\operatorname{div} \mathbf{v}\right) \mathbb{I} \right) - \rho^\star \frac{\mathrm{d}\overline{\lambda}}{\mathrm{d}t^\star} \mathbb{I}$$

$$\rho^\star \frac{\mathrm{Ra}^{\frac{2}{3}}}{\mathrm{Ra}_{\mathrm{crit}}^{\frac{1}{6}}} \frac{\mathrm{d}\vartheta^\star}{\mathrm{d}t^\star} = \Delta^\star \vartheta^\star + \frac{\Lambda_3}{\Lambda_2} \frac{\rho^\star}{2} \operatorname{Tr} \overline{\mathbb{S}}^\star + \frac{2}{3} \Lambda_3 \Lambda_1 \left(\vartheta^\star + \frac{\theta_{\mathrm{ref}}}{\theta_{\mathrm{diff}}} \right) \left[\mathbb{D}^\star : \overline{\mathbb{S}}^\star + \frac{\rho^\star}{\Lambda_2} \left(\overline{\lambda} \operatorname{div}^\star \mathbf{v}^\star - \frac{1}{2} \operatorname{Tr} \overline{\mathbb{S}}^\star \right) \right] + \Lambda_3 \mathrm{Ra}^{\frac{1}{3}} \mathrm{Ra}_{\mathrm{crit}}^{\frac{1}{6}} \left(\vartheta^\star + \frac{\theta_{\mathrm{ref}}}{\theta_{\mathrm{diff}}} \right) \frac{\mathrm{d}m^\star}{\mathrm{d}t^\star}$$

Parameter values

Parameter	Definition	Value
Pr	$\frac{\eta_{1,\mathrm{ref}}}{\rho_{\mathrm{ref}}\, k_{\mathrm{ref}}}$	4.0×10^{23}
Ra	$\frac{\rho_{\mathrm{ref}}\, g\, \alpha_{\mathrm{ref}}\, \theta_{\mathrm{diff}}\, l_{\mathrm{char}}^3}{k_{\mathrm{ref}}\, \eta_{1,\mathrm{ref}}}$	5.6×10^{4}
Λ_1	$\alpha_{\mathrm{ref}}\theta_{\mathrm{diff}}$	1.4×10^{-2}
Λ_2	$\frac{\tau_{\mathrm{ref}}}{t_{\mathrm{char}}}$	1.5×10^{-4}
Λ_3	$\frac{\eta_{1,\mathrm{ref}}\, l_{\mathrm{char}}^2}{\kappa_{\mathrm{ref}}\, \theta_{\mathrm{diff}}\, t_{\mathrm{char}}^2}$	5.1×10^{-1}

Table: Dimensionless parameters for convection in the Earth's mantle.

Conclusion

- Thermodynamically consistent model for mechanically incompressible Maxwell type fluids that are thermally compressible or expansible.
- Identification of the full system of the governing equations. (No Oberbeck–Boussinesq type approximation.)
- Discussion of the validity of the Oberbeck–Boussinesq type approximation in experiments/real problems involving viscoelastic fluids.

V. Průša and K. R. Rajagopal. On models for viscoelastic materials that are mechanically incompressible and thermally compressible or expansible and their Oberbeck–Boussinesq type approximations. *Math. Models Meth. Appl. Sci.*, 23(10):1761–1794, 2013