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RESEARCH NOTE

Wavefield and static deformation in depth-dependent elastic models with seismic sources of finite dimensions: spectral approach

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SUMMARY

A new formulation is presented to describe both the wavefield and the static coseismic deformation in depth-dependent elastic models generated by an *a priori* prescribed plane dislocation seismic source of finite dimensions, which is represented by an arbitrary time-dependent slip function changeable in both spatial dimensions along the fault. Elastic moduli can also change with depth in the source depth range. By employing a special Cartesian decomposition of displacement and continuity of traction acting on the fault, the partial differential equations of motion are converted into a set of ordinary differential equations over the depth for the displacement–stress vector, where the horizontal wavenumbers and the frequency play the role of parameters and the slip function is transformed into the source term of the equations. The resultant formulae thus represent a set of 1-D boundary-value problems, where no Green functions are needed. The pre-stress terms are considered in the equations of motion to obtain correct static limit.

Key words: 1-D elastic models, seismic source, Somigliana dislocation, waves and static response.

1 MOTIVATION

Most earthquakes are caused by an abrupt slip varying both in space and time (Somigliana dislocation) along faults. In continuum mechanics such a seismic source can be modelled by prescribing a time-dependent discontinuity of the displacement vector and keeping simultaneously continuity of the traction vector on the fault. To model co-seismic deformations, the static responses to final slips have to be evaluated.

In solid-state physics static problems of this kind had been studied already 100 years ago—see pp. 221–228 in Love (1944), who commented on work by Volterra (1907) on the problem of static deformation of a hollow cylinder after removal of a thin slice and subsequent joining of the plane faces so formed. Love added: “I have ventured to call them ‘dislocations’.” One of the first researchers, who realized the importance of the theory of dislocations in geophysics, was Stekete. In his paper (Stekete 1958) he dealt with the slip function of the type of a rigid-body displacement of finite dimensions in a homogeneous elastic half-space to obtain the tool for a quantitative description of fracture zones.

In simple models the static response to dislocations can be directly obtained by means of analytical formulae. In the effort to obtain analytical formulae special attention has been paid to static deformation of homogeneous half-spaces (see, e.g., the review in Okada 1992). The problem with such analytical formulae is that they may be oversimplified in particular applications because of the simplicity of the medium and thus other methods for more complex elastic models are required. Employing the Thomson–Haskell matrix method, Singh (1970) described the surface static response of an isotropic multilayered half-space to 3-D sources in the cylindrical system of vector functions and Singh & Garg (1985) presented integral representation of the surface displacements caused by a line source and discussed possible extension of their formulae to inner points as well as to models with a finite fault source. Their results were extended to the case of transversally isotropic layered half-space by Pan (1989). The subsurface deformations were then studied in detail by Roth (1990). A 2-D discontinuity method in multilayered media having irregular interfaces was presented by Dahm (1996). His approach is based on integral representation of displacements on an interface, where either tractions or displacements at one side of the boundary may be prescribed, which can be applied, for example, to problems where the stress drop is assumed to be known *a priori*.

In order to model generation of seismic waves, the dislocation boundary conditions on a fault are traditionally replaced by body-force equivalents (e.g. Aki & Richards 1980; Dahlen & Tromp 1998). The response of an elastic medium to such a source is then given by means of Green functions. In simple models the response can be again directly obtained by means of analytical formulae—see, e.g., Aki & Richards (1980) for the well-known expression of the response of a homogeneous infinite medium to a point source or Rybicki (1986) for a review of Green function methods both in static and dynamic geophysical problems. However, analytical formulae obtained for the homogeneous medium cannot usually be employed in more complicated models. This is the reason why in many cases, where a fully 3-D model of a studied region is not required or known, the Green function formalism for finite seismic sources is employed, although only a depth dependence of elastic moduli is taken into account. However, it is not necessary to use this formalism in such situations because the elastic equations of motion can be reduced by transform techniques to a set of first-order differential equations. For example, Kennett & Kerry (1979) used a Fourier–Bessel transform to describe seismic waves in a stratified elastic half-space. They considered a point source, which was determined by the moment tensor, and represented it by means of a discontinuity of the displacement–stress vector in the matrix propagator technique. The source of finite dimensions in this kind of model can be introduced into the propagator technique by decomposition of the wavefield into the waves directly radiated from the source and into reverberated waves coming from other depths. Propagation of the reverberated waves can be described by means of standard propagators and the directly radiated waves are obtained by solving the momentum equation in a uniform layer containing the source (see the review by Spudich & Archuleta 1987).

The aim of this note is to express the elastic response of 1-D (depth-dependent) models to a general dislocation source of finite dimensions with an arbitrary dip angle of the fault and arbitrary slip function without employing the body-force equivalent of such a source and thus without dealing with Green functions. Instead, the Fourier transform over horizontal coordinates is employed. However, location of a fault plane with finite dimensions inside the medium does not enable one to convert directly the horizontal part of the corresponding differential operators into algebraic expressions. This is the reason why a special decomposition of displacement satisfying boundary conditions on the fault is proposed. The application of the Fourier transform then results in a set of ordinary differential equations for the displacement–stress vector over depth, where horizontal wavenumbers and frequency only play the role of parameters and the slip function is directly converted into the source term of the equations. The obtained ordinary differential equations can then be solved by standard numerical methods, e.g. by finite differences with pseudospectral accuracy on the Bessel grids (Fornberg 1996) between the interfaces with jumps of elastic parameters, where the momentum equation is replaced by the requirement of continuity of the displacement vector and the traction vector, see also Hanyk *et al.* (2002), where this approach was successfully used in numerical experiments describing a viscoelastic response to surface loading.

2 METHOD

2.1 Decomposition of displacement

We will represent the displacement $\mathbf{u}$ by means of the spectral decomposition

$$\mathbf{u}(x,y,z,t) = \frac{1}{\sqrt{(2\pi)^3}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{V}(k_x,k_y,\omega;z)e^{i(k_x x+k_y y+\omega t)}\,dk_x\,dk_y\,d\omega$$
$$+\,e^{-[\operatorname{sgn}(x+z\cot\delta)](x+z\cot\delta)/L}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\mathbf{C}(k_y,\omega;z)+H(x+z\cot\delta)\mathbf{D}(k_y,\omega;z)\right]e^{i(k_y y+\omega t)}\,dk_y\,d\omega, \quad (1)$$

where (see Fig. 1) x, y are the horizontal coordinates chosen in such a way that the y-axis lies in the fault plane; z is the depth; δ is the dip angle; sgn is the signum function; $x + z\cot\delta = 0$ is the equation determining the fault plane; L is a characteristic length (e.g. the length of the fault); H is the Heaviside function; $\mathbf{D}$ is the representation of the slip on the fault in the $(k_y, \omega; z)$-domain. In forward problems we suppose that $\mathbf{D}$ is prescribed *a priori* and that $\mathbf{D} = 0$ outside a rectangle $\langle y_1, y_2\rangle \times \langle z_1, z_2\rangle$, $-\infty < y_1 < y_2 < \infty$, $0 < z_1 < z_2 < \infty$, after transforming $\mathbf{D}(k_y, \omega; z)$ back into the $(\omega; y, z)$ domain. $\mathbf{C}$ is an auxiliary smooth function, which will be determined so that the traction acting at the fault plane should be continuous. $\mathbf{V}$ is a smooth function, which will be determined so that the whole displacement should satisfy the momentum equation and boundary conditions.

Let us consider in the $(k_y, \omega; x, z)$-domain the displacement

$$\mathbf{W}(k_y,\omega;x,z) = e^{-[\operatorname{sgn}(x+z\cot\delta)](x+z\cot\delta)/L}\left[\mathbf{C}(k_y,\omega;z)+H(x+z\cot\delta)\mathbf{D}(k_y,\omega;z)\right] \quad (2)$$

and introduce

$$\operatorname{sgn}(s) = \operatorname{sgn}(x+z\cot\delta) \quad H(s) = H(x+z\cot\delta). \quad (3)$$

Now we will study the stress induced by the displacement $\mathbf{W}$ in the subdomains outside the fault, i.e. for $x + z\cot\delta \neq 0$. In the $(k_y, \omega; x, z)$-domain we can write

$$\nabla\cdot\mathbf{W} = -\frac{\operatorname{sgn}(s)}{L}W_x + ik_yW_y - \frac{\operatorname{sgn}(s)}{L}\cot\delta\,W_z + e^{-\operatorname{sgn}(s)(x+z\cot\delta)/L}\frac{\partial}{\partial z}[C_z + H(s)D_z]. \quad (4)$$

The corresponding stress tensor is

$$\tau_{xx}(\mathbf{W}) = \lambda(z)\nabla\cdot\mathbf{W} - 2\mu(z)\frac{\operatorname{sgn}(s)}{L}W_x, \quad (5)$$

$$\tau_{yy}(\mathbf{W}) = \lambda(z)\nabla\cdot\mathbf{W} + 2\mu(z)ik_yW_y, \quad (6)$$

$$\tau_{zz}(\mathbf{W}) = \lambda(z)\nabla\cdot\mathbf{W} + 2\mu(z)\left\{-\frac{\operatorname{sgn}(s)\cot\delta}{L}W_z + e^{-\operatorname{sgn}(s)(x+z\cot\delta)/L}\frac{\partial}{\partial z}[C_z + H(s)D_z]\right\}, \quad (7)$$

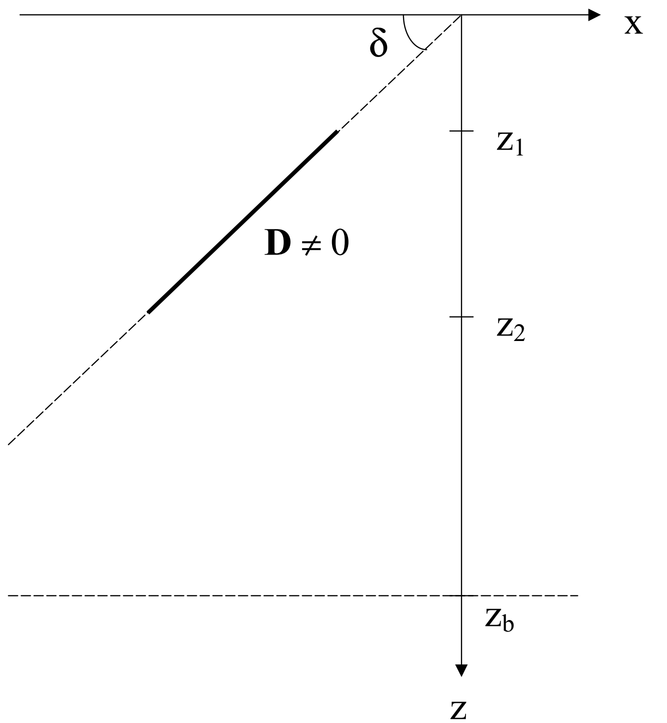

Figure 1. Description of the model and the choice of the coordinates. The solid line in the depth interval $\langle z_1, z_2\rangle$ represents an active part of the fault with non-zero slip, the dashed line shows the remaining inactive part of the fault plane. The active part of the fault is finite in the third y-dimension and the slip is also y-dependent. The fault is inside the layer of thickness z_b with depth-dependent Lamé coefficients and the layer lies on a homogeneous half-space.

$$\tau_{xy}(\mathbf{W}) = \mu(z)\left[ik_y W_x - \frac{\operatorname{sgn}(s)}{L}W_y\right], \tag{8}$$

$$\tau_{xz}(\mathbf{W}) = \mu(z)\left\{-\frac{\operatorname{sgn}(s)\cot\delta}{L}W_x + \mathrm{e}^{-\operatorname{sgn}(s)(x+z\cot\delta)/L}\frac{\partial}{\partial z}[C_x + H(s)D_x] - \frac{\operatorname{sgn}(s)}{L}W_z\right\}, \tag{9}$$

$$\tau_{yz}(\mathbf{W}) = \mu(z)\left\{-\frac{\operatorname{sgn}(s)\cot\delta}{L}W_y + \mathrm{e}^{-\operatorname{sgn}(s)(x+z\cot\delta)/L}\frac{\partial}{\partial z}[C_y + H(s)D_y] + ik_y W_z\right\}, \tag{10}$$

where $\lambda(z)$ and $\mu(z)$ are the depth-dependent Lamé coefficients. As the normal to the fault plane is $\mathbf{n} = (\sin\delta, 0, \cos\delta)$, we can express the traction force vector $\mathbf{T}$ acting at the fault plane, where $\mathrm{e}^{-\operatorname{sgn}(s)(x+z\cot\delta)/L} \to 1$, in the form

$$\begin{aligned}T_x =& \lambda\sin\delta\left[-\frac{\operatorname{sgn}(s)}{L}\{C_x + H(s)D_x + \cot\delta[C_z + H(s)D_z]\} + ik_y[C_y + H(s)D_y] + \frac{\partial}{\partial z}[C_z + H(s)D_z]\right]\\ &- 2\mu\sin\delta\frac{\operatorname{sgn}(s)}{L}[C_x + H(s)D_x] + \mu\cos\delta\left[-\frac{\operatorname{sgn}(s)}{L}\{\cot\delta[C_x + H(s)D_x] + C_z + H(s)D_z\} + \frac{\partial}{\partial z}[C_x + H(s)D_x]\right],\end{aligned} \tag{11}$$

$$\begin{aligned}T_y =& \mu\left[\sin\delta\left\{ik_y[C_x + H(s)D_x] - \frac{\operatorname{sgn}(s)}{L}[C_y + H(s)D_y]\right\}\right.\\ &\left.+\cos\delta\left\{-\frac{\operatorname{sgn}(s)}{L}\cot\delta[C_y + H(s)D_y] + \frac{\partial}{\partial z}[C_y + H(s)D_y] + ik_y[C_z + H(s)D_z]\right\}\right],\end{aligned} \tag{12}$$

$$\begin{aligned}T_z =& \lambda\cos\delta\left[-\frac{\operatorname{sgn}(s)}{L}\{C_x + H(s)D_x + \cot\delta[C_z + H(s)D_z]\} + ik_y[C_y + H(s)D_y] + \frac{\partial}{\partial z}[C_z + H(s)D_z]\right]\\ &+ 2\mu\cos\delta\left\{\frac{\partial}{\partial z}[C_z + H(s)D_z] - \frac{\operatorname{sgn}(s)}{L}\cot\delta[C_z + H(s)D_z]\right\}\\ &+\mu\sin\delta\left[\frac{\partial}{\partial z}[C_x + H(s)D_x] - \frac{\operatorname{sgn}(s)}{L}\{\cot\delta[C_x + H(s)D_x] + C_z + H(s)D_z\}\right].\end{aligned} \tag{13}$$

Continuity of **T** across the fault plane, i.e. $\mathbf{T}(s>0)-\mathbf{T}(s<0)=0$ for $|s|\to 0$, then yields the equations determining **C**,

$$0=\lambda\sin\delta\left\{-\frac{1}{L}[2C_x+D_x+\cot\delta(2C_z+D_z)]+ik_yD_y+\frac{\partial}{\partial z}D_z\right\}-2\mu\frac{\sin\delta}{L}(2C_x+D_x)$$
$$+\mu\cos\delta\left\{-\frac{1}{L}[\cot\delta(2C_x+D_x)+2C_z+D_z]+\frac{\partial}{\partial z}D_x\right\}, \tag{14}$$

$$0=\sin\delta\left[ik_yD_x-\frac{1}{L}(2C_y+D_y)\right]+\cos\delta\left[-\frac{\cot\delta}{L}(2C_y+D_y)+\frac{\partial}{\partial z}D_y+ik_yD_z\right], \tag{15}$$

$$0=\lambda\cos\delta\left\{-\frac{1}{L}[2C_x+D_x+\cot\delta(2C_z+D_z)]+ik_yD_y+\frac{\partial}{\partial z}D_z\right\}+2\mu\cos\delta\left[\frac{\partial}{\partial z}D_z-\frac{\cot\delta}{L}(2C_z+D_z)\right]$$
$$+\mu\sin\delta\left\{\frac{\partial}{\partial z}D_x-\frac{1}{L}[\cot\delta(2C_x+D_x)+2C_z+D_z]\right\}. \tag{16}$$

The solution of this system is

$$C_y=\frac{L}{2(\sin\delta+\cos\delta\cot\delta)}\left[\sin\delta\left(ik_yD_x-\frac{D_y}{L}\right)+\cos\delta\left(\frac{\partial}{\partial z}D_y-\frac{\cot\delta}{L}D_y+ik_yD_z\right)\right], \tag{17}$$

$$C_x=\frac{f_xa_{zz}-f_za_{xz}}{a_{xx}a_{zz}-a_{zx}a_{xz}}, \tag{18}$$

$$C_z=\frac{f_za_{xx}-f_xa_{zx}}{a_{xx}a_{zz}-a_{zx}a_{xz}}, \tag{19}$$

where

$$a_{xx}=(\lambda+2\mu)\sin\delta+\mu\cos\delta\cot\delta, \tag{20}$$

$$a_{xz}=(\lambda+\mu)\cos\delta, \tag{21}$$

$$a_{zx}=(\lambda+\mu)\cos\delta, \tag{22}$$

$$a_{zz}=(\lambda+2\mu)\cos\delta\cot\delta+\mu\sin\delta, \tag{23}$$

$$f_x=\frac{L}{2}\left\{\lambda\sin\delta\left[-\frac{1}{L}(D_x+\cot\delta D_z)+ik_yD_y+\frac{\partial}{\partial z}D_z\right]-\frac{2}{L}\mu\sin\delta D_x+\mu\cos\delta\left[-\frac{1}{L}(D_x\cot\delta+D_z)+\frac{\partial}{\partial z}D_x\right]\right\}, \tag{24}$$

$$f_z=\frac{L}{2}\left\{\lambda\cos\delta\left[-\frac{1}{L}(D_x+\cot\delta D_z)+ik_yD_y+\frac{\partial}{\partial z}D_z\right]+2\mu\cos\delta\left(\frac{\partial}{\partial z}D_z-\frac{\cot\delta}{L}D_z\right)+\mu\sin\delta\left[-\frac{1}{L}(D_x\cot\delta+D_z)+\frac{\partial}{\partial z}D_x\right]\right\}. \tag{25}$$

The vertical fault ($\delta=90^\circ$) represents the special simpler case, which yields

$$C_x=\frac{L\lambda}{2(\lambda+2\mu)}\left(-\frac{1}{L}D_x+ik_yD_y+\frac{\partial}{\partial z}D_z\right)-\frac{\mu}{\lambda+2\mu}D_x, \tag{26}$$

$$C_y=ik_y\frac{L}{2}D_x-\frac{1}{2}D_y, \tag{27}$$

$$C_z=\frac{L}{2}\frac{\partial}{\partial z}D_x-\frac{1}{2}D_z. \tag{28}$$

The simplest relation between **C** and **D** can be obtained for $\delta\to 0^\circ$:

$$\mathbf{C}=-\frac{1}{2}\mathbf{D}. \tag{29}$$

Now we will deal with the following six-component vector:

$$\mathbf{Y}(\mathbf{W})(k_y,\omega;x,z)=\begin{pmatrix}W_x\\W_y\\W_z\\\tau_{xz}(\mathbf{W})\\\tau_{yz}(\mathbf{W})\\\tau_{zz}(\mathbf{W})\end{pmatrix}=\begin{pmatrix}(C_x+H(s)D_x)\mathrm{e}^{-\mathrm{sgn}(s)z\cot\delta/L}\\(C_y+H(s)D_y)\mathrm{e}^{-\mathrm{sgn}(s)z\cot\delta/L}\\(C_z+H(s)D_z)\mathrm{e}^{-\mathrm{sgn}(s)z\cot\delta/L}\\\tau_{xz}(\mathbf{W})\mathrm{e}^{+\mathrm{sgn}(s)x/L}\\\tau_{yz}(\mathbf{W})\mathrm{e}^{+\mathrm{sgn}(s)x/L}\\\tau_{zz}(\mathbf{W})\mathrm{e}^{+\mathrm{sgn}(s)x/L}\end{pmatrix}\mathrm{e}^{-\mathrm{sgn}(s)x/L}\equiv\mathbf{E}^{(s)}(k_y,\omega;z)\mathrm{e}^{-\mathrm{sgn}(s)x/L}. \tag{30}$$

As we will write the momentum equations for $\mathbf{V}$ in the $(k_x, k_y, \omega; z)$-domain, we introduce $\mathbf{Y}(\mathbf{W})(k_x, k_y, \omega; z)$ by the relation

$$\mathbf{Y}(\mathbf{W})(k_y, \omega; x, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{Y}(\mathbf{W})(k_x, k_y, \omega; z) e^{ik_x x}\, dk_x \tag{31}$$

and thus

$$\begin{aligned}\mathbf{Y}(\mathbf{W})(k_x, k_y, \omega; z) &= \frac{1}{\sqrt{2\pi}} \left[\mathbf{E}^{(s>0)}(k_y, \omega; z) \int_{-z\cot\delta}^{\infty} e^{-(ik_x+1/L)x}\, dx + \mathbf{E}^{(s<0)}(k_y, \omega; z) \int_{-\infty}^{-z\cot\delta} e^{-(ik_x-1/L)x}\, dx \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\mathbf{E}^{(s>0)}(k_y, \omega; z) \frac{e^{(ik_x+1/L)z\cot\delta}}{ik_x + 1/L} + \mathbf{E}^{(s<0)}(k_y, \omega; z) \frac{e^{(ik_x-1/L)z\cot\delta}}{-ik_x + 1/L} \right].\end{aligned} \tag{32}$$

2.2 Governing equations in the $(k_x, k_y, \omega; z)$-domain

In the $(k_x, k_y, \omega; z)$-domain we can now easily express Hooke's law for the whole displacement $(\mathbf{V} + \mathbf{W})(k_x, k_y, \omega; z)$ as

$$\nabla \cdot (\mathbf{V} + \mathbf{W}) = ik_x(V_x + W_x) + ik_y(V_y + W_z) + \frac{\partial}{\partial z}(V_z + W_z) \tag{33}$$

and the corresponding stress tensor is

$$\tau_{xx}(\mathbf{V} + \mathbf{W}) = \lambda(z)\nabla \cdot (\mathbf{V} + \mathbf{W}) + 2\mu(z) ik_x(V_x + W_x), \tag{34}$$

$$\tau_{yy}(\mathbf{V} + \mathbf{W}) = \lambda(z)\nabla \cdot (\mathbf{V} + \mathbf{W}) + 2\mu(z) ik_y(V_y + W_y), \tag{35}$$

$$\tau_{zz}(\mathbf{V} + \mathbf{W}) = \lambda(z)\nabla \cdot (\mathbf{V} + \mathbf{W}) + 2\mu(z) \frac{\partial}{\partial z}(V_z + W_z), \tag{36}$$

$$\tau_{xy}(\mathbf{V} + \mathbf{W}) = \mu(z)[ik_y(V_x + W_x) + ik_x(V_y + W_y)], \tag{37}$$

$$\tau_{xz}(\mathbf{V} + \mathbf{W}) = \mu(z)\left[\frac{\partial}{\partial z}(V_x + W_x) + ik_x(V_z + W_z)\right], \tag{38}$$

$$\tau_{yz}(\mathbf{V} + \mathbf{W}) = \mu(z)\left[\frac{\partial}{\partial z}(V_y + W_y) + ik_y(V_z + W_z)\right]. \tag{39}$$

In the following formulae we will use the notation $\mathbf{Y} \equiv \mathbf{Y}(\mathbf{V}) + \mathbf{Y}(\mathbf{W})$. Hooke's law as expressed above yields

$$\frac{\partial Y_1}{\partial z} + ik_x Y_3 - \frac{1}{\mu} Y_4 = 0, \tag{40}$$

$$\frac{\partial Y_2}{\partial z} + ik_y Y_3 - \frac{1}{\mu} Y_5 = 0, \tag{41}$$

$$\frac{\partial Y_3}{\partial z} + \frac{ik_x\lambda}{\lambda + 2\mu} Y_1 + \frac{ik_y\lambda}{\lambda + 2\mu} Y_2 - \frac{1}{\lambda + 2\mu} Y_6 = 0. \tag{42}$$

In order to obtain the correct formulae in the $\omega \to 0$ limit (coseismic deformation), we will consider the momentum equation including also the pre-stress terms:

$$\nabla \cdot \boldsymbol{\tau}(\boldsymbol{u}) + \left(\rho_0 \frac{dg_0}{dz} u_z - \rho_0 g_0 \nabla \cdot \mathbf{u}\right) \mathbf{e}_z = \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2}, \tag{43}$$

where $\rho_0 \equiv \rho_0(z)$ is the density before deformation, $g_0 \equiv g_0(z)$ is the corresponding gravity acceleration and $\mathbf{e}_z$ is the vertical unit vector pointing downward. In the $(k_x, k_y, \omega; z)$-domain this equation can be rewritten as the three scalar equations,

$$\lambda\left(-k_x^2 Y_1 - k_x k_y Y_2 + ik_x \frac{\partial Y_3}{\partial z}\right) - 2\mu k_x^2 Y_1 - \mu\left(k_y^2 Y_1 + k_x k_y Y_2\right) + \frac{\partial Y_4}{\partial z} + \rho_0 \omega^2 Y_1 = 0, \tag{44}$$

$$\lambda\left(-k_x k_y Y_1 - k_y^2 Y_2 + ik_y \frac{\partial Y_3}{\partial z}\right) - 2\mu k_y^2 Y_2 - \mu\left(k_x k_y Y_1 + k_x^2 Y_2\right) + \frac{\partial Y_5}{\partial z} + \rho_0 \omega^2 Y_2 = 0, \tag{45}$$

$$ik_x Y_4 + ik_y Y_5 + \frac{\partial Y_6}{\partial z} + \rho_0 \frac{dg_0}{dz} Y_3 - \rho_0 g_0 \left(ik_x Y_1 + ik_y Y_2 + \frac{\partial Y_3}{\partial z}\right) + \rho_0 \omega^2 Y_3 = 0. \tag{46}$$

Now we will substitute $\partial Y_3/\partial z$ and rearrange this system as follows:

$$\frac{\partial Y_4}{\partial z} + \left[-(\lambda + 2\mu)k_x^2 - \mu k_y^2 + \frac{\lambda^2}{\lambda + 2\mu} k_x^2 + \rho_0 \omega^2\right] Y_1 + \left[-(\lambda + \mu) + \frac{\lambda^2}{\lambda + 2\mu}\right] k_x k_y Y_2 + \frac{ik_x\lambda}{\lambda + 2\mu} Y_6 = 0, \tag{47}$$

$$\frac{\partial Y_5}{\partial z} + \left[-(\lambda + \mu) + \frac{\lambda^2}{\lambda + 2\mu}\right] k_x k_y Y_1 + \left[-(\lambda + 2\mu)k_y^2 - \mu k_x^2 + \frac{\lambda^2}{\lambda + 2\mu} k_y^2 + \rho_0 \omega^2\right] Y_2 + \frac{ik_y\lambda}{\lambda + 2\mu} Y_6 = 0, \tag{48}$$

$$\frac{\partial Y_6}{\partial z} - \rho_0 g_0 \frac{2\mu}{\lambda+2\mu}(ik_x Y_1 + ik_y Y_2) + \left(\rho_0 \frac{dg_0}{dz} + \rho_0\omega^2\right) Y_3 + ik_x Y_4 + ik_y Y_5 - \frac{\rho_0 g_0}{\lambda+2\mu} Y_6 = 0. \tag{49}$$

We have thus arrived at the final system of equations:

$$\frac{\partial}{\partial z}\mathbf{Y}(\mathbf{V}) + \mathbf{A}\mathbf{Y}(\mathbf{V}) = -\frac{\partial}{\partial z}\mathbf{Y}(\mathbf{W}) - \mathbf{A}\mathbf{Y}(\mathbf{W}), \tag{50}$$

where the matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & ik_x & -\frac{1}{\mu} & 0 & 0 \\ 0 & 0 & ik_y & 0 & -\frac{1}{\mu} & 0 \\ \frac{ik_x\lambda}{\lambda+2\mu} & \frac{ik_y\lambda}{\lambda+2\mu} & 0 & 0 & 0 & -\frac{1}{\lambda+2\mu} \\ -4\mu\frac{\lambda+\mu}{\lambda+2\mu}k_x^2 - \mu k_y^2 + \rho_0\omega^2 & \left[-(\lambda+\mu) + \frac{\lambda^2}{\lambda+2\mu}\right]k_x k_y & 0 & 0 & 0 & \frac{ik_x\lambda}{\lambda+2\mu} \\ \left[-(\lambda+\mu) + \frac{\lambda^2}{\lambda+2\mu}\right]k_x k_y & -4\mu\frac{\lambda+\mu}{\lambda+2\mu}k_y^2 - \mu k_x^2 + \rho_0\omega^2 & 0 & 0 & 0 & \frac{ik_y\lambda}{\lambda+2\mu} \\ -\rho_0 g_0\frac{2\mu}{\lambda+2\mu}ik_x & -\rho_0 g_0\frac{2\mu}{\lambda+2\mu}ik_y & \rho_0\frac{dg_0}{dz} + \rho_0\omega^2 & ik_x & ik_y & -\frac{\rho_0 g_0}{\lambda+2\mu} \end{pmatrix}. \tag{51}$$

2.3 Decomposition into toroidal and spheroidal parts

If we introduce the toroidal part of the displacement and the traction,

$$\mathbf{Y}_T = (ik_x Y_2 - ik_y Y_1, ik_x Y_5 - ik_y Y_4), \tag{52}$$

as well as their spheroidal part,

$$\mathbf{Y}_S = (ik_x Y_1 + ik_y Y_2, ikY_3, ik_x Y_4 + ik_y Y_5, ikY_6), \quad k \equiv \sqrt{k_x^2 + k_y^2}, \tag{53}$$

the final system of equations from the previous section can be replaced by the decoupled system

$$\frac{\partial}{\partial z}\mathbf{Y}_T(\mathbf{V}) + \mathbf{A}_T\mathbf{Y}(\mathbf{V}) = -\frac{\partial}{\partial z}\mathbf{Y}_T(\mathbf{W}) - \mathbf{A}_T\mathbf{Y}(\mathbf{W}), \tag{54}$$

$$\frac{\partial}{\partial z}\mathbf{Y}_S(\mathbf{V}) + \mathbf{A}_S\mathbf{Y}(\mathbf{V}) = -\frac{\partial}{\partial z}\mathbf{Y}_S(\mathbf{W}) - \mathbf{A}_S\mathbf{Y}(\mathbf{W}), \tag{55}$$

where the matrices $\mathbf{A}_T$ and $\mathbf{A}_S$ are

$$\mathbf{A}_T = \begin{pmatrix} 0 & -\frac{1}{\mu} \\ -\mu k^2 + \rho_0\omega^2 & 0 \end{pmatrix}, \tag{56}$$

$$\mathbf{A}_S = \begin{pmatrix} 0 & ik & -\frac{1}{\mu} & 0 \\ ik\frac{\lambda}{\lambda+2\mu} & 0 & 0 & -\frac{1}{\lambda+2\mu} \\ -4k^2\mu\frac{\lambda+\mu}{\lambda+2\mu} + \rho_0\omega^2 & 0 & 0 & ik\frac{\lambda}{\lambda+2\mu} \\ -ik\rho_0 g_0\frac{2\mu}{\lambda+2\mu} & \rho_0\frac{dg_0}{dz} + \rho_0\omega^2 & ik & -\frac{\rho_0 g_0}{\lambda+2\mu} \end{pmatrix}, \tag{57}$$

2.4 Boundary conditions

If there is a free surface at $z = 0$, the corresponding boundary conditions are

$$Y_4(\mathbf{V}+\mathbf{W}) = Y_5(\mathbf{V}+\mathbf{W}) = Y_6(\mathbf{V}+\mathbf{W}) = 0. \tag{58}$$

Moreover, we will consider a homogeneous half-space for $z > z_b > 0$, i.e. λ and μ are positive constants for $z > z_b$. We will also neglect the influence of pre-stress in this half-space, which can be done formally by setting $g_0 = 0$ in $\mathbf{A}_S$, and no source will be placed here, i.e. $\mathbf{W}(z) = 0$ for $z > z_b$. Now we will express the solution of the momentum equations in the half-space analytically following the matrix representation, which can be found in Aki & Richards (1980).

The eigenvalues of $\mathbf{A}_T$ are $\mp\sqrt{k^2-\beta^{-2}\omega^2}$, where $\beta=\sqrt{\mu/\rho_0}$ is the velocity of *S* waves, and the corresponding eigenvectors form the columns of the matrix

$$\mathbf{E}_T=\begin{pmatrix}1 & 1\\ \mu\sqrt{k^2-\beta^{-2}\omega^2} & -\mu\sqrt{k^2-\beta^{-2}\omega^2}\end{pmatrix}. \tag{59}$$

The solution can thus be written in the form

$$\mathbf{Y}_T=\mathbf{E}_T\mathbf{L}_T\mathbf{I}_T, \tag{60}$$

where

$$\mathbf{L}_T(z)=\begin{pmatrix}e^{\sqrt{k^2-\beta^{-2}\omega^2}(z-z_\mathrm{b})} & 0\\ 0 & e^{-\sqrt{k^2-\beta^{-2}\omega^2}(z-z_\mathrm{b})}\end{pmatrix}. \tag{61}$$

and $\mathbf{I}_T$ is the vector of integration constants. If $k^2\le\beta^{-2}\omega^2$, the solution describes upgoing and downgoing *SH* waves; if $k^2>\beta^{-2}\omega^2$, we obtain diverging and converging Love waves. As the source is located at shallower depth, no upgoing *SH* wave can be generated. Similarly, we will consider only Love waves with zero amplitude at infinity. These requirements can be simply satisfied by putting

$$I_{T1}=0. \tag{62}$$

Since

$$\mathbf{L}_T^{-1}(z)\mathbf{E}_T^{-1}=\begin{pmatrix}e^{-\sqrt{k^2-\beta^{-2}\omega^2}(z-z_\mathrm{b})} & 0\\ 0 & e^{\sqrt{k^2-\beta^{-2}\omega^2}(z-z_\mathrm{b})}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \frac{1}{2\mu\sqrt{k^2-\beta^{-2}\omega^2}}\\ \frac{1}{2} & -\frac{1}{2\mu\sqrt{k^2-\beta^{-2}\omega^2}}\end{pmatrix} \tag{63}$$

and

$$\mathbf{I}_T=\mathbf{L}_T^{-1}(z_\mathrm{b})\mathbf{E}_T^{-1}\mathbf{Y}_T=\mathbf{E}_T^{-1}\mathbf{Y}_T, \tag{64}$$

the condition $I_{T1}=0$ then yields the boundary condition at $z=z_\mathrm{b}$:

$$\frac{1}{2}Y_{T1}+\frac{1}{2\mu\sqrt{k^2-\beta^{-2}\omega^2}}Y_{T2}=0. \tag{65}$$

Analogously, the eigenvalues of $\mathbf{A}_S$ are $\mp\sqrt{k^2-\alpha^{-2}\omega^2}$, $\mp\sqrt{k^2-\beta^{-2}\omega^2}$, where $\alpha=\sqrt{(\lambda+2\mu)/\rho_0}$ is the velocity of *P* waves, and the vector $\mathbf{Y}_S$ describes the *P–SV* waves and/or the Rayleigh waves. The corresponding matrices $\mathbf{E}_S$ and $\mathbf{L}_S$ are

$$\mathbf{E}_S=\begin{pmatrix}\frac{\alpha k}{\omega} & \frac{\alpha k}{\omega} & -i\frac{\beta\sqrt{k^2-\beta^{-2}\omega^2}}{\omega} & -i\frac{\beta\sqrt{k^2-\beta^{-2}\omega^2}}{\omega}\\ -i\frac{\alpha\sqrt{k^2-\alpha^{-2}\omega^2}}{\omega} & i\frac{\alpha\sqrt{k^2-\alpha^{-2}\omega^2}}{\omega} & -\frac{\beta k}{\omega} & \frac{\beta k}{\omega}\\ 2\frac{\rho_0\alpha\beta^2k\sqrt{k^2-\alpha^{-2}\omega^2}}{\omega} & -2\frac{\rho_0\alpha\beta^2k\sqrt{k^2-\alpha^{-2}\omega^2}}{\omega} & i\omega\rho_0\beta\left(1-2\beta^2\frac{k^2}{\omega^2}\right) & -i\omega\rho_0\beta\left(1-2\beta^2\frac{k^2}{\omega^2}\right)\\ i\omega\rho_0\alpha\left(1-2\beta^2\frac{k^2}{\omega^2}\right) & i\omega\rho_0\alpha\left(1-2\beta^2\frac{k^2}{\omega^2}\right) & -2\frac{\rho_0\beta^3k\sqrt{k^2-\beta^{-2}\omega^2}}{\omega} & -2\frac{\rho_0\beta^3k\sqrt{k^2-\beta^{-2}\omega^2}}{\omega}\end{pmatrix}, \tag{66}$$

$$\mathbf{L}_S(z)=\begin{pmatrix}e^{\sqrt{k^2-\alpha^{-2}\omega^2}(z-z_\mathrm{b})} & 0 & 0 & 0\\ 0 & e^{-\sqrt{k^2-\alpha^{-2}\omega^2}(z-z_\mathrm{b})} & 0 & 0\\ 0 & 0 & e^{\sqrt{k^2-\beta^{-2}\omega^2}(z-z_\mathrm{b})} & 0\\ 0 & 0 & 0 & e^{-\sqrt{k^2-\beta^{-2}\omega^2}(z-z_\mathrm{b})}\end{pmatrix}. \tag{67}$$

Therefore, the boundary conditions for these waves are

$$I_{S1}=I_{S3}=0. \tag{68}$$

Since

$$\mathbf{E}_S^{-1}=\begin{pmatrix}\frac{\beta^2k}{\alpha\omega} & \frac{1-2\beta^2(k^2/\omega^2)}{2\alpha\sqrt{k^2-\alpha^{-2}\omega^2}}i\omega & \frac{k}{2\omega\rho_0\alpha\sqrt{k^2-\alpha^{-2}\omega^2}} & \frac{-i}{2\omega\rho_0\alpha}\\ \frac{\beta^2k}{\alpha\omega} & -\frac{1-2\beta^2(k^2/\omega^2)}{2\alpha\sqrt{k^2-\alpha^{-2}\omega^2}}i\omega & -\frac{k}{2\omega\rho_0\alpha\sqrt{k^2-\alpha^{-2}\omega^2}} & \frac{-i}{2\omega\rho_0\alpha}\\ \frac{1-2\beta^2(k^2/\omega^2)}{2\beta\sqrt{k^2-\beta^{-2}\omega^2}}i\omega & -\frac{\beta k}{\omega} & \frac{-i}{2\omega\rho_0\beta} & -\frac{k}{2\omega\rho_0\beta\sqrt{k^2-\beta^{-2}\omega^2}}\\ \frac{1-2\beta^2(k^2/\omega^2)}{2\beta\sqrt{k^2-\beta^{-2}\omega^2}}i\omega & \frac{\beta k}{\omega} & \frac{i}{2\omega\rho_0\beta} & -\frac{k}{2\omega\rho_0\beta\sqrt{k^2-\beta^{-2}\omega^2}}\end{pmatrix}, \tag{69}$$

we finally obtain the boundary conditions at $z = z_b$ in the form

$$\frac{\beta^2 k}{\alpha\omega} Y_{S1} + \frac{1 - 2\beta^2 (k^2/\omega^2)}{2\alpha\sqrt{k^2 - \alpha^{-2}\omega^2}} i\omega Y_{S2} + \frac{k}{2\omega\rho_0\alpha\sqrt{k^2 - \alpha^{-2}\omega^2}} Y_{S3} - \frac{i}{2\omega\rho_0\alpha} Y_{S4} = 0, \tag{70}$$

$$\frac{1 - 2\beta^2 (k^2/\omega^2)}{2\beta\sqrt{k^2 - \beta^{-2}\omega^2}} i\omega Y_{S1} - \frac{\beta k}{\omega} Y_{S2} + \frac{-i}{2\omega\rho_0\beta} Y_{S3} - \frac{k}{2\omega\rho_0\beta\sqrt{k^2 - \beta^{-2}\omega^2}} Y_{S4} = 0. \tag{71}$$

2.5 Static response

We will obtain the static response by performing $\omega \to 0$ limit in the formulae presented above. There is no problem with toroidal deformation. The matrix $\mathbf{A}_T$ has the form

$$\mathbf{A}_T = \begin{pmatrix} 0 & -\frac{1}{\mu} \\ -\mu k^2 & 0 \end{pmatrix}, \tag{72}$$

and the corresponding boundary condition at $z = z_b$ is

$$\mu k Y_{T1} + Y_{T2} = 0. \tag{73}$$

Spheroidal deformation is more complicated. The matrix $\mathbf{A}_S$ is easily obtained again by setting $\omega = 0$,

$$\mathbf{A}_S = \begin{pmatrix} 0 & ik & -\frac{1}{\mu} & 0 \\ ik\frac{\lambda}{\lambda + 2\mu} & 0 & 0 & -\frac{1}{\lambda + 2\mu} \\ -4k^2\mu\frac{\lambda + \mu}{\lambda + 2\mu} & 0 & 0 & ik\frac{\lambda}{\lambda + 2\mu} \\ -ik\rho_0 g_0 \frac{2\mu}{\lambda + 2\mu} & \rho_0 \frac{dg_0}{dz} & ik & -\frac{\rho_0 g_0}{\lambda + 2\mu} \end{pmatrix}. \tag{74}$$

The boundary conditions (70) and (71), however, become linearly dependent for $\omega \to 0$. The reason is that there are only two eigenvalues of $\mathbf{A}_S$ in the homogeneous half-space, $\mp k$, and only two linearly independent eigenvectors

$$\boldsymbol{F}^{\mp} = \begin{pmatrix} 1 \\ \mp i \\ \pm 2\mu k \\ -i2\mu k \end{pmatrix}. \tag{75}$$

Nevertheless, there are two generalized eigenvectors

$$\boldsymbol{G}^{\mp} = \begin{pmatrix} \pm\frac{\lambda + 2\mu}{k(\lambda + \mu)} + \frac{i}{k} \\ \pm 1k + i\frac{\mu}{k(\lambda + \mu)} \\ 2\mu \pm i2\mu \\ 2\mu \end{pmatrix} \tag{76}$$

satisfying the equation

$$\mathbf{A}_S \boldsymbol{G}^{\mp} = \mp k \boldsymbol{G}^{\mp} - \boldsymbol{F}^{\mp}. \tag{77}$$

Then the four independent solutions in the half-space are

$$\boldsymbol{F}^{\mp} \mathrm{e}^{\pm k(z - z_b)}, \quad [\boldsymbol{G}^{\mp} + z\boldsymbol{F}^{\mp}] \mathrm{e}^{\pm k(z - z_b)}. \tag{78}$$

The non-diverging solution in the half-space can thus be written in the form

$$\mathbf{Y}_S = I_1 \boldsymbol{F}^{+} \mathrm{e}^{-k(z - z_b)} + I_2 (\boldsymbol{G}^{+} + z\boldsymbol{F}^{+}) \mathrm{e}^{-k(z - z_b)}, \tag{79}$$

where I_1 and I_2 are integration constants. At $z = z_b$, eq. (79) represents the four boundary equations for the six unknowns: the four components of the solution vector $\mathbf{Y}_S$ and the two integration constants I_1 and I_2. We can express the integration constants, e.g. from the first and the second equation of (79) to obtain

$$I_1 = Y_{S1} + \frac{-\lambda - 2\mu + k(\lambda + \mu)z_b + i(\lambda + \mu)}{\lambda + 3\mu}(Y_{S1} + iY_{S2}), \qquad I_2 = -\frac{k(\lambda + \mu)}{\lambda + 3\mu}(Y_{S1} + iY_{S2}). \tag{80}$$

After putting these expressions of the integration constants into eq. (79) the third and the fourth equations of eq. (79) are then the required boundary conditions. The degeneracy of the static limit in the matrix method applied to layered models was analysed by Zhu & Rivera (2002).

3 CONCLUDING REMARK

The studied problem has been converted to the systems of ordinary differential equations over depth for the vector of unknowns consisting of the displacement components and the horizontal traction components with the boundary conditions both at the surface and at the top of the homogeneous layer below the zone of interest. No Green functions were used as we worked directly with the slip function determining Somigliana dislocation.

The horizontal wavenumbers as well as the frequency play the role of parameters in each system and thus in horizontal dimensions the problem can be discretized by means of the discrete wavenumber method, which is simply explained in Bouchon (2003); see also the original papers by Bouchon & Aki (1977) and Bouchon (1979). Such a discretization was shown to be an effective tool in computations of the stress field radiated by the six elements of the moment tensor in plane-layered media (Cotton & Coutant 1997).

Vertical discretization can be performed, e.g. by finite differences with pseudospectral accuracy (Fornberg 1996; Hanyk *et al.* 2002). From the numerical point of view, this approach is more straightforward than the discretization by means of a system of homogeneous layers and subsequent employment of the matrix method, which yields rather complicated formulae both in wavefield (Kennett 2001) and static (Roth 1990) calculations. As the fault is of finite dimensions and the Lamé coefficients can vary with depth also along the fault, the presented approach enables, in principle, to deal with situations, where the fault penetrates a material interface, which can result in unexpected effects (Bonafede *et al.* 2002; Rivalta *et al.* 2002).

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