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On the solvability of the Stokes pseudo-boundary-value problem for geoid determination

Z. Martinec, C. Matyska

Department of Geophysics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 180 00 Prague 8, Czech Republic

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Abstract A new form of boundary condition of the Stokes problem for geoid determination is derived. It has an unusual form, because it contains the unknown disturbing potential referred to both the Earth's surface and the geoid coupled by the topographical height. This is a consequence of the fact that the boundary condition utilizes the surface gravity data that has not been continued from the Earth's surface to the geoid. To emphasize the 'two-boundary' character, this boundary-value problem is called the Stokes pseudo-boundary-value problem. The numerical analysis of this problem has revealed that the solution cannot be guaranteed for all wavelengths. We demonstrate that geoidal wavelengths shorter than some critical finite value must be excluded from the solution in order to ensure its existence and stability. This critical wavelength is, for instance, about 1 arcmin for the highest regions of the Earth's surface.

Furthermore, we discuss various approaches frequently used in geodesy to convert the 'two-boundary' condition to a 'one-boundary' condition only, relating to the Earth's surface or the geoid. We show that, whereas the solution of the Stokes pseudo-boundary-value problem need not exist for geoidal wavelengths shorter than a critical wavelength of finite length, the solutions of approximately transformed boundary-value problems exist over a larger range of geoidal wavelengths. Hence, such regularizations change the nature of the original problem; namely, they define geoidal heights even for the wavelengths for which the original Stokes pseudo-boundary-value problem need not be solvable.

1. Introduction

The traditional formulation of the problem for geoid height determination over a continental area with non-zero terrain elevations comes from the fact that heights of the terrain above the geoid and the modulus of the surface gravity are known with a certain accuracy (Vaníček and Kleusberg, 1987; Sideris and Forsberg, 1991). The effort is to employ Stokes's integral for determining the disturbing potential of the geoid. To be able to use this integral, two requirements must be satisfied: that there are no masses outside the geoid, and that the gravity observations are referred to the geoid. One possibility, which satisfies these requirements only approximately, however, is as follows: The observed modulus of gravity on the Earth's surface that is a non-linear function of the gravity potential is linearized with respect to a reference gravity potential. The linearized boundary functional, after removing a large part of the gravitational attraction of topographical masses by a compensation technique, is continued from the Earth's surface to the geoid. Due to linearization, the geoid radius in the disturbing potential may be approximated by the radius of a sphere approaching the Earth's surface; Stokes's classical boundary-value problem is thus formulated and solved on the geoid. Finally, the gravitational effect of the topographical masses is restored.

It is a common belief that, after removing the first-degree spherical harmonics from the gravitational potential, only a regularization of the downward continuation of a high frequency part of the gravity is necessary to guarantee the existence of a unique solution of the Stokes problem for geoid determination. In this paper, we will deal with the original formulation of the problem prior to the downward continuation of gravity. We intend to demonstrate that, besides the spherical harmonics of degree one, the existence of the solution is not also guaranteed for higher-degree harmonics. As far as we know, this lack of an additional guarantee has not been studied yet; the solution may not exist due to the fact that the input data — the surface gravity and the potential of the geoid — are prescribed on different boundaries.

Correspondence to: Z. Martinec

2. Formulation of the problem

Let the geocentric radius of the geoid be described by an angularly dependent function $r = r_g(\Omega)$, where (r, Ω) are the geocentric spherical coordinates, i.e., $(r_g(\Omega), \Omega)$ are points lying on the geoid. We will assume that the function $r_g(\Omega)$ is not known. Let $H(\Omega)$ be the height of the Earth's surface above the geoid reckoned along the geocentric radius. Unlike the geocentric radius of the geoid, we will assume that $H(\Omega)$ is a known function. Let the masses between the geoid and the Earth's surface be termed the topographical masses. We will assume that there are no masses outside the Earth's surface. Finally, let the following quantities be given: the gravity $g_S(\Omega)$ measured on the Earth's surface, the density $\varrho(r, \Omega)$ of the topographical masses, the gauge value W_0 of the gravity potential on the geoid, and the angular velocity ω of the Earth's rotation.

The question we are dealing with is how to determine the gravity potential $W(r, \Omega)$ inside and outside the topographical masses and the radius $r_g(\Omega)$ of the geoid. The problem is governed by Poisson's equation with the boundary conditions given on the free boundaries coupled by means of height $H(\Omega)$, the geoid, and the Earth's surface:

$$\nabla^2 W = -4\pi G \varrho + 2\omega^2 \quad \text{outside}\, (r_g(\Omega), \Omega)\,, \tag{1}$$

$$|\, \text{grad} W\, (r_g(\Omega) + H(\Omega), \Omega)| = g_S(\Omega)\,, \tag{2}$$

$$W(r_g(\Omega), \Omega) = W_0\,, \tag{3}$$

$$W = \frac{1}{2}\omega^2 r^2 \sin^2\vartheta + \frac{GM}{r} + O\left(\frac{1}{r^3}\right) \quad r \to \infty\,, \tag{4}$$

where GM is a geocentric constant, ϑ is geocentric co-latitude, and ϱ is equal to zero outside the Earth's surface $(r_g(\Omega) + H(\Omega), \Omega)$.

3. Compensation of topographical masses

To be able to work with a harmonic gravitational potential outside the geoid, it is necessary to replace the topographical masses (extending between the geoid and the surface) with an auxiliary body situated below or on the geoid and to find the changing the gravitational potential due to this replacement. To start with, let us have a look at the gravitational potential V^t induced by the topographical masses:

$$V^t(r, \Omega) = G \int_{\Omega_0} \int_{r'=r_g(\Omega')}^{r_g(\Omega')+H(\Omega')} \frac{\varrho(r', \Omega')}{L(r, \psi, r')} r'^2 dr' d\Omega'\,, \tag{5}$$

where Ω_0 is the full solid angle, and $L(r, \psi, r')$ is the distance between the computation point (r, Ω) and an integration point (r', Ω'). The notation $L(r, \psi, r')$ is used to emphasize the fact that L depends only on radial distances r, and r' and the angle ψ between the radii of the points. It is a well-known fact that the topographical masses generate a strong gravitational field with equipotential surfaces undulating by hundreds of metres with respect to a level ellipsoid. Since the undulations of the geoid are significantly smaller, there must exist compensation mechanisms of topographical masses which reduce their gravitational effect. These mechanisms are probably associated mainly with lateral heterogeneities of the crust but also partly with deep dynamical processes (Matyska, 1994). To describe this compensation mathematically, a number of more or less idealized compensation models have been proposed. For the purpose of geoid computation, we may, in principle, employ any compensation model generating a harmonic gravitational field outside the geoid. For instance, the topographic-isostatic compensation models (e.g., Rummel et al., 1988; Moritz, 1990) are based on compensation by the anomalies of density distribution $\varrho_c(r, \Omega)$ in a layer between the geoid and the compensation level $r_c(\Omega), r_c(\Omega) < r_g(\Omega)$, i.e., the gravitational potential

$$V^{isost.}(r, \Omega) = G \int_{\Omega_0} \int_{r'=r_c(\Omega')}^{r_g(\Omega')} \frac{\varrho_c(r', \Omega')}{L(r, \psi, r')} r'^2 dr' d\Omega' \tag{6}$$

reduces the gravitational effect of topographical masses. Although the way of compensation might originally be motivated by geophysical ideas, it is, from the point of view of our problem, only an auxiliary mathematical operation. We must only express the change in the potential due to this operation to be able to compute the geoid of the original body.

In the limiting case, the topographical masses may be compensated by a mass surface located on the geoid, i.e., by a layer whose thickness is infinitely small. This kind of compensation, called Helmert's 2nd condensation (Helmert, 1884), is described by Newton's surface integral:

$$V^{conden.}(r, \Omega) = G \int_{\Omega_0} \frac{\sigma(\Omega')}{L(r, \psi, r_g(\Omega'))} r_g^2(\Omega') d\Omega'\,, \tag{7}$$

where $\sigma(\Omega)$ is the density of a condensation layer; the density $\sigma(\Omega)$ may be chosen by various ways depending on the manner of approximation used for fitting the topographical potential V^t by the condensation potential $V^{conden.}$ (Martinec, 1993). There is an open question, not solved here, whether Helmert's 2nd condensation technique, popular recently (e.g., Martinec et al., 1993), is the best way to compensate the gravitational effect of topographical masses, or whether other types of compensation of topographical masses, e.g., isostatic compensations, stabilize the solution of the Stokes boundary-value problem in a more efficient way.

Having introduced a compensation mechanism of topographical masses, the associated compensation potential V^c approximating the topographical potential V^t reads

$$V^c = V^{isost.} \quad \text{or} \quad V^c = V^{conden.} \tag{8}$$

for the respective isostatic compensation and Helmert's condensation of topographical masses. Finally, let us introduce the *residual topographical potential* δV,

$$\delta V = V^t - V^c\,, \tag{9}$$

which is the rest of the fit of V^t by V^c.

4. Disturbing potential

The gravity potential W can be considered as a sum of the gravitational potential V^g generated by the masses below the geoid, the topographical potential V^t, and the centrifugal potential V^Q:

$$W = V^g + V^t + V^Q. \tag{10}$$

Inserting from eqn. (9) into (10) for potential V^t, the gravity potential W becomes

$$W = V^g + V^c + \delta V + V^Q \ . \tag{11}$$

Now, let us decompose the gravity potential $V^g + V^c + V^Q$ into the sum of the (known) normal gravity potential U generated by a level ellipsoid spinning with the same angular velocity as the Earth and a (unknown) disturbing potential T^h:

$$V^g + V^c + V^Q = U + T^h. \tag{12}$$

The superscript 'h' emphasizes that T^h approaches the actual disturbing potential T of the Earth, $T = W - U$. The difference between T and T^h is given by the residual topographical potential δV:

$$\delta V = T - T^h. \tag{13}$$

The problem (1)-(4) expressed in terms of the disturbing potential T^h reads:

$$\bigtriangledown^2 T^h = 0 \quad \text{outside}\,(r_g(\Omega), \Omega), \tag{14}$$

$$|\,\text{grad}(U + T^h + \delta V)(r_g(\Omega) + H(\Omega), \Omega)| = g_s(\Omega) \tag{15}$$

$$(U + T^h + \delta V)(r_g(\Omega), \Omega) = W_0\,, \tag{16}$$

together with the asymptotic condition at infinity,

$$T^h(r, \Omega) = O\left(\frac{1}{r^3}\right) \quad r \to \infty\,. \tag{17}$$

The last condition also shows that the zero- and first-degree spherical harmonics have been excluded from the disturbing potential T^h. This can be achieved by a suitable choice of normal gravity potential U.

5. Brun's formula

Let P, P_g, and Q be points on the Earth's surface, the geoid, and the reference ellipsoid, respectively. Let points P and P_g lie on the same geocentric radius line and point Q be the so-called normal point to P_g. Point Q is established such that (i) the normal gravity potential U at Q is equal to the actual gravity potential W_0 at P_g, and (ii) P_g and Q lie on the same plumb line of the normal gravity field.

Using this notation, the boundary condition (16) may be written as

$$U_{P_g} + T^h_{P_g} + \delta V_{P_g} = W_0\,. \tag{18}$$

The normal potential U_{P_g} may be expressed by means of U and its derivatives at point Q:

$$U_{P_g} = U_Q + \left.\frac{\partial U}{\partial h}\right|_Q N + \ldots \tag{19}$$

$$\doteq U_Q - \gamma_Q N \ , \tag{20}$$

where $\partial/\partial h$ is the derivative along the plumb line of the normal gravity field; γ is the normal gravity, $\gamma = |\text{gradU}|$; and N is the height of the geoid above the reference ellipsoid. The higher-order terms of the Taylor series expansion (19) that have been neglected make an error (in geoidal heights) of 10^{-3}m maximally (Vaníček and Martinec, 1994). Assuming that $U_Q = W_0$ and substituting eqn.(20) into (18), we get Bruns's formula for the geoidal height N in the form:

$$\boxed{N = \tfrac{1}{\gamma_Q}(T^h + \delta V)|_{P_g}\,.} \tag{21}$$

The term $T^h_{P_g}/\gamma_Q$ yields the undulations of the so-called **co-geoid** with respect to the reference ellipsoid. Note that the co-geoid is the equipotential surface of the gravity potential $V^g + V^c + V^Q$ with the gauge value W_0. The term $\delta V_{P_g}/\gamma_Q$ yields the undulations of the geoid with respect to the co-geoid; it is termed the primary indirect topographical effect on the geoid (Heck, 1993; Martinec and Vaníček, 1994a).

6. Linearization of the boundary condition

Let us apply the operator grad $\{\cdot\}$ to the gravity potential $W = U + T^h + \delta V$ and take the magnitude of the resulting vector. We get

$$|\text{grad}W| = |\text{grad}U| + \frac{\text{grad}U \cdot \text{grad}(T^h + \delta V)}{|\text{grad}U|} + O\left(\frac{|\text{grad}(T^h + \delta V)|^2}{|\text{grad}U|}\right), \tag{22}$$

where '.' denotes the scalar product of vectors. Neglecting the last term in eqn.(22) and taking a spherical approximation of the second term on the right-hand side corrected to the flattening of the Earth, we approximately get

$$g \doteq \gamma - \frac{\partial T^h}{\partial r} - \frac{\partial \delta V}{\partial r} + \epsilon_h(T^h) + \epsilon_h(\delta V)\,, \tag{23}$$

where $g = |\text{grad}W|$ is the actual gravity, and the ellipsoidal correction term $\epsilon_h(T)$ is given by (e.g., Jekeli, 1981, eqn. (4.15))

$$\epsilon_h(T) = \alpha \sin 2\vartheta \frac{1}{r}\frac{\partial T}{\partial \vartheta}; \tag{24}$$

where α is the flattening of the Earth. Note that eqn. (23) is valid everywhere above the geoid.

Particularly, let us consider eqn. (23) at a point P on the topographical surface,

$$\left.\frac{\partial T^h}{\partial r}\right|_P - \epsilon_h(T^h_P) = -g_P + \gamma_P - \delta A_P\,, \tag{25}$$

where

$$\delta A_P = \frac{\partial \delta V}{\partial r}\bigg|_P - \epsilon_h(\delta V_P) \tag{26}$$

is the direct topographical effect on gravity (Heck, 1993; Martinec and Vaníček, 1994b).

Normal gravity γ_P may be expressed by means of γ and its derivatives at point Q:

$$\gamma_P = \gamma_Q + \frac{\partial \gamma}{\partial h}\bigg|_Q N + \frac{\partial \gamma}{\partial r}\bigg|_Q H + \frac{1}{2}\frac{\partial^2 \gamma}{\partial r^2}\bigg|_Q H^2 + \dots \ , \tag{27}$$

where the magnitude of terms neglected does not exceed 0.01 mGal. Introducing the free-air reduction

$$F = -\frac{\partial \gamma}{\partial r}\bigg|_Q H - \frac{1}{2}\frac{\partial^2 \gamma}{\partial r^2}\bigg|_Q H^2 - \dots \ , \tag{28}$$

eqn. (27) reads

$$\gamma_P = \gamma_Q - F + \frac{\partial \gamma}{\partial h}\bigg|_Q N \ . \tag{29}$$

The derivative of the normal gravity γ along the plumb line of the normal gravity field can be approximately expressed as (e.g., Cruz, 1985, eqn. (2.22))

$$\frac{\partial \gamma}{\partial h} = -\frac{2\gamma}{r} + \frac{2\gamma}{r}\alpha(3\cos^2\vartheta - 2) \ . \tag{30}$$

Substituting eqns. (29) and (30) into eqn. (25) and expressing the geoidal height N by Bruns's formula (21), we get the final form of the linearized boundary condition for the disturbing potential T^h:

$$\boxed{\frac{\partial T^h}{\partial r}\bigg|_P + \frac{2}{r_Q} T^h_{P_g} - \epsilon_h(T^h_P) - \epsilon_\gamma(T^h_{P_g}) = -\Delta g^h \ ,} \tag{31}$$

where the gravity anomaly Δg^h consists of terms

$$\Delta g^h = \Delta g^F + \delta A_P + \delta_s \ . \tag{32}$$

Here we have introduced the free-air gravity anomaly Δg^F,

$$\Delta g^F = g_P - \gamma_Q + F \ , \tag{33}$$

the ellipsoidal correction term $\epsilon_\gamma(T)$,

$$\epsilon_\gamma(T) = 2\alpha\,(3\cos^2\vartheta - 2)\frac{T}{r_Q} \ , \tag{34}$$

and the secondary indirect effect on gravity δ_s (Wichiencharoen, 1982),

$$\delta_s = -\frac{\partial \gamma}{\partial h}\bigg|_Q \frac{\delta V_{P_g}}{\gamma_Q} \doteq \frac{2}{r_Q}\delta V_{P_g} - \epsilon_\gamma(\delta V_{P_g}) \ . \tag{35}$$

Inspecting boundary condition (31), we can see that the term $\partial T^h/\partial r$ is referred to the Earth's surface, whereas the potential T^h in the term $2T^h/r$ is referred to the geoid. Hence, eqn.(31) represents a non-standard boundary condition with the unknown referred to the two boundaries, the geoid and the Earth's surface, coupled by height $H(\Omega)$. Adopting the terminology introduced by Sansò (1995), such a problem can be classified as the pseudo-boundary-value problem; here we will call it the *Stokes pseudo-boundary-value problem*. This terminology emphasizes that instead of having the boundary condition relating to the geoid, the boundary condition (31) contains a mixture of unknown T^h referred to the geoid as well as to the Earth's surface.

The original problem (1)–(4) as well as the problem described by eqns. (14), (17), (21) with the linearized boundary condition (31) are scalar non-linear free boundary-value problems since the radial coordinate of the geoid is one of the unknowns to be determined. Having some approximation of the geoid, it is very easy to transform the latter free boundary-value problem into a problem with fixed boundaries. For example, replacing P_g with r_Q and P with $r_Q + H$ in eqn. (31) yields the ellipsoidal approximation of the Stokes pseudo-boundary-value problem, where eqns. (14), (17) and the modified boundary condition (31) serve to determine T^h; eqn. (21) then gives the geoidal height N. Another possibility, most often used in geoid height computations, is to approximate the geoid in the boundary condition (31) by a mean sphere with radius $R = 6371$ km. This means that the radius of point P_g is replaced by R and the radius of point P by $R + H(\Omega)$. The relative error introduced by this spherical approximation is of the order of 3×10^{-3} in the classical problems (Heiskanen and Moritz, 1967, sect. 2-14), which then causes a long-wavelength error of at most 0.5 metres in geoidal heights. In regional problems, where only shorter wavelengths are to be determined, this approximation is often reasonable. In the following numerical tests we will employ the spherical approximation of boundary condition (31) for its simplicity. We intend to concentrate on the effects connected with the 'two-boundary nature' of this condition that appear only in a very short wavelength part of the solution.

7. Numerical investigations

In this section we will solve the Stokes pseudo-boundary-value problem numerically for the cases when function $H(\Omega)$ takes special forms. Our numerical test will be focused on demonstrating that, in general, the solution of the Stokes pseudo-boundary-value problem does not exist. We will attempt to find the restrictions ensuring the existence of the solution.

The solution of the Laplace equation (14) with the condition (17) is

$$T^h(r,\Omega) = \sum_{j=j_{min}}^{j_{max}} \sum_{m=-j}^{j} T_{jm}\left(\frac{R}{r}\right)^{j+1} Y_{jm}(\Omega) \ , \tag{36}$$

where $j_{min}(\geq 2)$ and j_{max} are the respective minimum and maximum cut-off degrees, and T_{jm} are the coefficients of the potential T^h to be determined. In order to normalize the potential coefficients T_{jm}, we have introduced the mean Earth's radius R into the expansion (36). Eqn. (31) in the spherical approximation then becomes

$$\frac{1}{R}\sum_{j=j_{min}}^{j_{max}}\sum_{m=-j}^{j}\left[(j+1)\left(\frac{R}{R+H(\Omega)}\right)^{j+2}-2 +2\alpha(3\cos^2\vartheta-2)\right]Y_{jm}(\Omega)T_{jm} + \frac{\alpha}{R}\sum_{j=j_{min}}^{j_{max}}\sum_{m=-j}^{j}\left(\frac{R}{R+H(\Omega)}\right)^{j+2}\sin 2\vartheta\frac{\partial Y_{jm}(\Omega)}{\partial\vartheta}T_{jm} = \Delta g^h . \quad (37)$$

This boundary condition must hold in any direction Ω. In order to ensure this, we will employ the Galerkin method (Lapidus and Pinder, 1982) in which eqn. (37) can be rewritten as a system of linear algebraic equations for coefficients T_{jm}:

$$\boldsymbol{Am} = \boldsymbol{d} \ , \quad (38)$$

where $\boldsymbol{m}$ is a column vector composed of potential coefficients T_{jm}, i.e.,

$$\boldsymbol{m} = \{T_{jm}|\ j = j_{min}, \ldots, j_{max}, m = -j, \ldots, j\} \ , \quad (39)$$

$\boldsymbol{A}$ is the matrix composed of the weighted left-hand side of eqn. (37),

$$A_{j_1m_1,jm} = \int_{\Omega_0}\left[(j+1)\left(\frac{R}{R+H(\Omega)}\right)^{j+2}-2+2\alpha(3\cos^2\vartheta-2)\right] Y_{jm}(\Omega)Y^*_{j_1m_1}(\Omega)d\Omega + \alpha\int_{\Omega_0}\left(\frac{R}{R+H(\Omega)}\right)^{j+2} \sin 2\vartheta\frac{\partial Y_{jm}(\Omega)}{\partial\vartheta}Y^*_{j_1m_1}(\Omega)d\Omega, \quad (40)$$

Ω_0 is the full solid angle, $d\Omega = \sin\vartheta d\vartheta d\lambda$, and $\boldsymbol{d}$ is a column vector of the weighted right-hand side of eqn. (37),

$$d_{j_1m_1} = R\int_{\Omega_0}\Delta g^h(\Omega)Y^*_{j_1m_1}(\Omega)d\Omega \ . \quad (41)$$

7.1. An example: Constant height

Let us first consider a simple, but illustrative, case when $H = H_0 = const$ over the Earth, and $\alpha = 0$. Introducing function

$$K_j(H_0) = (j+1)\left(\frac{R}{R+H_0}\right)^{j+2}-2, \quad \text{for } j \geq 2 \ , \quad (42)$$

the transfer matrix $A_{j_1m_1,jm}$ between unknown parameters T_{jm} and the gravity anomalies on the right-hand side of eqn. (37) becomes $A_{j_1m_1,jm} = K_j(H_0)\delta_{jj_1}\delta_{mm_1}$ and thus

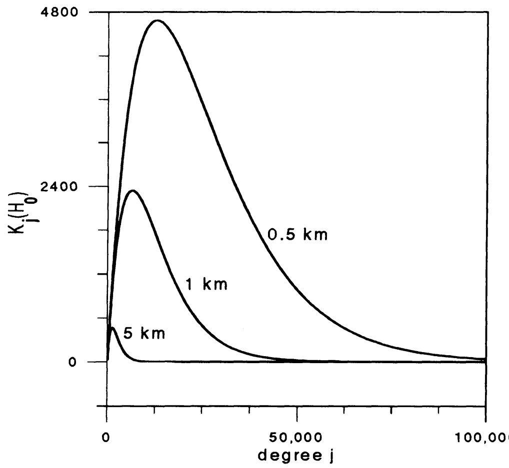

Fig. 1. Transfer function $K_j(H_0)$ between unknown coefficients T_{jm} and gravity anomalies Δg^h for $H_0 = 1$ km, 5 km, and 10 km

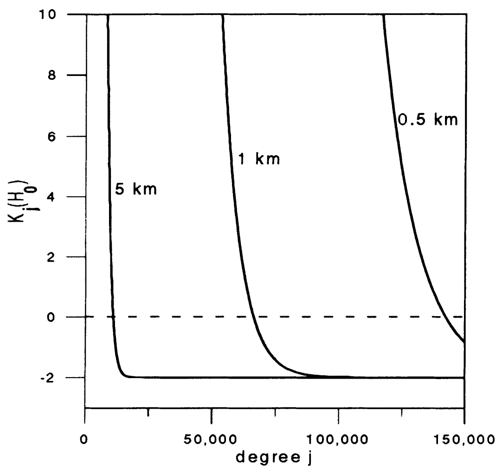

Fig. 1a. A detail of Fig. 1

$$T_{jm} = \frac{R}{K_j(H_0)}\int_{\Omega_0}\Delta g^h(\Omega)Y^*_{jm}(\Omega)d\Omega \ . \quad (43)$$

Since $0.998 < R/(R+H_0) < 1$ for the Earth, it is clear that $\lim_{j\to\infty} K_j = -2$ for any fixed $H_0 > 0$. On the other hand, $K_j > 0$ for low degrees j because $0.976 < K_2 < 1$. This means that there is a range of j's in which K_j is zero or near zero. For those j's the solution of eqns. (38) is unstable or even does not exist once $K_j = 0$.

Let us estimate the range of j's for which the solution of eqns. (38) becomes unstable for this simple example. Figure 1 plots the values of K_j for height H_0 equal to 1 km, 5 km and 10 km. We can see that the increase of K_j with increasing j is confined to low degrees j and then K_j starts to decrease to its limiting value -2. That is why the determination of disturbing potential T^h is stable only in some part of the spectral domain. The width of the stable part grows with decreasing H_0.

Figure 2 plots those j_{zero} for which function $K_j(H_0)$ vanishes. For such degrees matrix $\boldsymbol{A}$ is singular and the solution of the system of equations (38) does not exist. Since spherical degree j corresponds to a given resolution $\Delta\Omega$ in a spatial domain, $\Delta\Omega = \pi/j$, we may also convert critical degree j_{zero} to a critical spatial resolution size $\Delta\Omega_{zero}, \Delta\Omega_{zero} = \pi/j_{zero}$, for which the solution of our problem does not exist. Figure 2 shows that, for instance, $j_{zero} = 10\,980$, for $H_0 = 5$ km, and the critical spatial resolution size is $\Delta\Omega_{zero} \doteq 1$ arcmin. To interpret the result in other words, let us imagine that the Earth's topography is a Bouguer spherical plate with a constant height of 5 km above the geoid and the Stokes pseudo-boundary-value problem is solved in a spatial domain such that the potential $T^h(R, \Omega)$ is parameterized by discrete values $T^h(R, \Omega_i)$ is a regular angular grid with grid step size $\Delta\Omega$. Then the solution of the Stokes pseudo-boundary-value problem will not exist if the grid step size $\Delta\Omega$ of the parameterization of T^h is less than or equal to the critical step size $\Delta\Omega_{zero}$, i.e., of about 1 arcmin in our example, even though the surface gravity data would be known continuously on the Earth's surface.

To map the non-existence of the solution for regional geoid determination and for a more realistic model of the Earth's topography, we need to set up and solve the system of eqn.(38) for high degrees and orders ($j_{max} = 10^4 - 10^5$). This leads to computational difficulties because of the huge consummation of computational time and memory; with today's computer equipment it is impossible to carry out an analysis of the existence for such a general case. Thus, we are forced to approximate the Earth's surface by a simplified model of axisymmetric geometry. By making use the analysis of this simplified case, we will attempt to estimate the range of critical spectral degrees j_{zero} for the actual case.

7.2. Axisymmetric geometry

Let the height $H(\vartheta, \lambda)$ of the Earth's surface above the geoid is modelled by zonal as well as tesseral and sectoral spherical harmonics of the global digital terrain model TUG87 (Wieser, 1987) cut at degree 180. To create a rotational symmetric body, axisymmetric height

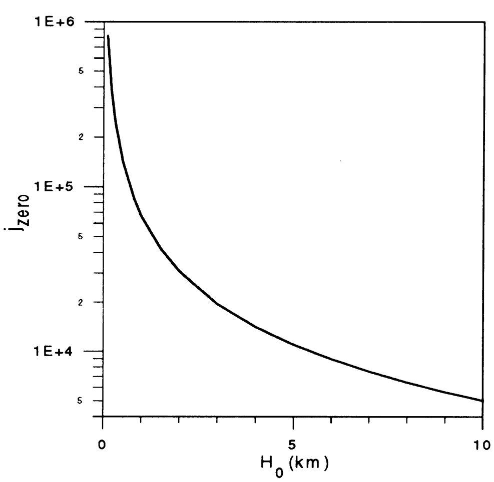

Fig. 2. The roots j_{zero} of function $K_j(H_0)$ for $H_0 \in (100\text{ m}, 10^4\text{ m})$

$H(\vartheta)$ will be generated by height $H(\vartheta, \lambda)$ taken along a fixed meridian $\lambda = \lambda_0$. In the case of an axisymmetric surface, the elements $A_{j_1 m_1, jm}$ of matrix $\boldsymbol{A}$ do not depend on angular orders m and m_1; they can be written as

$$A_{j_1 j} = \int_{\vartheta=0}^{\pi} \left[(j+1)\left(\frac{R}{R+H(\vartheta)}\right)^{j+2} - 2 + 2\alpha(3\cos^2\vartheta - 2) \right] P_j(\cos\vartheta) P_{j_1}(\cos\vartheta) \sin\vartheta d\vartheta + \alpha \int_{\vartheta=0}^{\pi} \left(\frac{R}{R+H(\vartheta)}\right)^{j+2} \sin 2\vartheta \frac{dP_j(\cos\vartheta)}{d\vartheta} \times P_{j_1}(\cos\vartheta) \sin\vartheta d\vartheta \ . \quad (44)$$

Note that the elements $A_{j_1 j}$ can only be evaluated by a method of numerical quadrature.

To analyse the posedness of the Stokes pseudo-boundary-value problem, we will employ the eigenvalue analysis of matrix $\boldsymbol{A}$. According to this method, a non-symmetric matrix $\boldsymbol{A}$ can be decomposed to the product of three matrices,

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1} \ , \quad (45)$$

where the columns of matrix $\boldsymbol{U}$ are formed from the right eigenvectors of $\boldsymbol{A}$, the rows of $\boldsymbol{U}^{-1}$ are formed from the left eigenvectors of $\boldsymbol{A}$, and the diagonal matrix $\boldsymbol{\Lambda}$ consists of eigenvalues of $\boldsymbol{A}$. We have employed subroutines BALANC, ELMHES, and HQR (Press et al., 1989) to find the eigenvalues of a non-symmetric matrix $\boldsymbol{A}$.

Figure 3 shows the topographical heights $H(\vartheta, \lambda_0)$ along the meridian profile $\lambda_0 = 80°$ reaching value $H_{max} = 5353$ metres. The consequent Fig. 4 shows a plot of the eigenvalues of matrix $\boldsymbol{A}$ for an axisymmetric body with the outer surfaces generated by this meridian profile. In order to avoid high degrees j, and thus, be able to perform the eigenvalue analysis in real CPU time, we multiply function $H(\vartheta)$ by a factor of 10. The minimum spherical degrees j_{min} of the potential series (36) is $j_{min} = 21$, which models the situation when low-degree harmonics of the potential T^h are determined by another approach, e.g., when considering a satellite gravitational model. In Fig. 4, where we further put the flattening of the Earth equal to zero, $\alpha = 0$, we change the maximum cut-off degree j_{max} of the disturbing

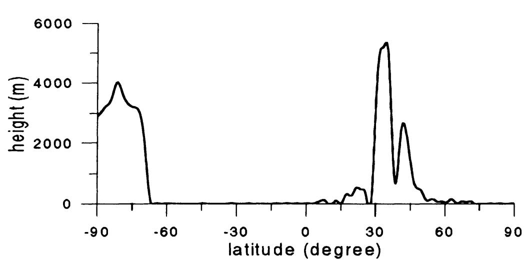

Fig. 3. The meridian profile $\lambda = 80°$ of topographical height $H(\vartheta, \lambda)$ generated by the global digital terrain model TUG87 (Wieser, 1987) cut at degree 180. This profile is used to create a body with the axisymmetric geometry of external surface

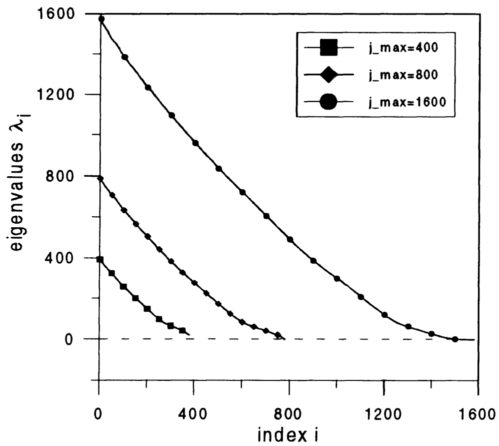

Fig. 4. The eigenvalue spectra of matrix A for various cut-off degrees j_{max} and a body with axisymmetric surface generated by height $H(\vartheta, \lambda = 80°)$ multiplied by 10 ($j_{min} = 21$). The ellipsoidal corrections ε_h and ε_γ are equal to zero

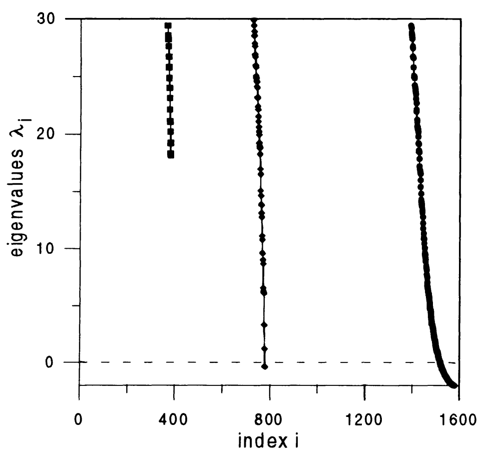

Fig. 4a. A detail of Fig. 4

potential T^h and plot eigenvalues of matrix A ordered according to their size (note that the eigenvalues are real numbers in this particular case). Inspecting Fig. 4 we can observe that the eigenvalue spectrum of matrix A intersects the zero level starting from degree $j_{zero} \doteq 800$. Once the cut-off degree j_{max} of the spherical harmonic expansion (36) of potential T^h is greater or equal to j_{zero}, the eigenvalue spectrum of A contains a null eigenvalue or eigenvalues of a very small size. The matrix A becomes ill-conditioned or even singular, and the inverse A^{-1} may be distorted by large round-off errors or may not exist at all; in such a case the Stokes pseudo-boundary-value problem does not have a unique and stable solution. As in the preceeding section, the critical degree j_{zero} can again be converted to the critical spatial discretization size $\Delta\Omega_{zero}$ for a case when the Stokes pseudo-boundary-value problem is solved in a spatial domain.

The next test investigates the influence of the ellipsoidal corrections terms ϵ_h and ϵ_γ on the posedness of matrix A. We choose the same body as in the preceeding example together with $j_{min} = 21$ and

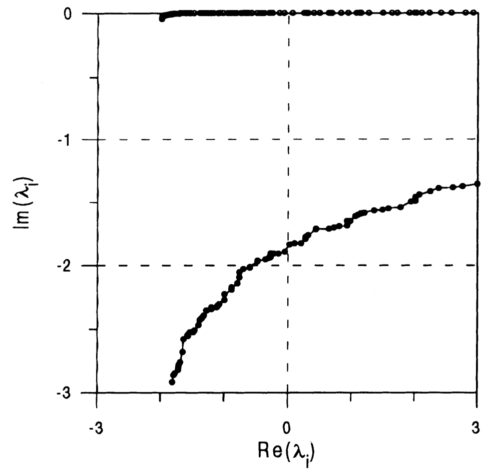

Fig. 5. The real vs. imaginary parts of the eigenvalues of matrix A with (upper branch) and without (lower branch) the ellipsoidal corrections ε_h and ε_γ. The axisymmetric body is the same as that considered in Fig. 4 ($j_{min} = 21$, and $j_{max} = 1600$)

$j_{max} = 1600$ and compute the eigenvalues of matrix A putting $\alpha = 0$ and $\alpha = 1/298.257$, respectively. Figure 5 shows those eigenvalues the magnitudes of which are smaller than 3. (Note that the eigenvalues of A for the case $\alpha = 1/298.257$ are complex numbers.) We can observe that the eigenvalue spectrum of A changes significantly when α differs from zero: there is no null eigenvalue and the magnitude of the smallest eigenvalue is larger than 1. In other words, the ellipsoidal corrections ϵ_γ and ϵ_h act as regularization factors removing the ill-posedness of matrix A. It also means that, in this particular case, ϵ_γ and ϵ_h cannot be subtracted from the right-hand side of eqn. (38) as known quantities determined a priorily by using a known global gravitational model of the Earth; such usage of ellipsoidal corrections is recommended in real geoid computation (e.g., Cruz, 1985).

In order to create a more realistic example, we use the same profile of topographical heights as plotted in Fig. 3, but now, in contrast with the preceeding example, we will not multiply heights $H(\vartheta)$ by 10. In this case, it is not possible to carry out the eigenvalue analysis of matrix A starting from degree $j_{min} = 21$ and going up to degrees $j_{max} \approx 10^4 - 10^5$ due to a huge consumption of computer time and memory. We have to confine ourselves to a smaller range of sought spherical harmonics. That is why we choose $j_{min} = 10\,000$ and j_{max} in the range between 10 300 and 11 600. The results for the case $\alpha = 0$ are shown in Fig. 6. We can again observe that eigenvalue spectra intersect the zero-level starting at degree $j_{zero} = 10\,500$. It means that whenever $j_{max} \geq j_{zero}$, the spectrum of matrix A contains an eigenvalue which is very close or equal to zero. Consequently, matrix A becomes ill-conditioned or even singular. Putting $\alpha = 1/298.257$ (this case is not plotted here) has a similar stabilization effect as in the case shown in Fig. 5.

To carry out the eigenvalue analysis of matrix A uses a lot of computer time. However, the critical spherical

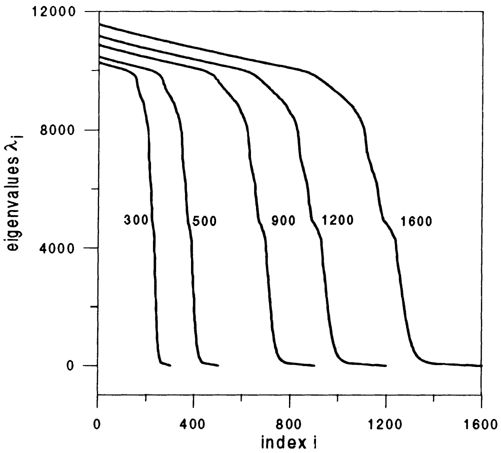

Fig. 6. The eigenvalue spectra of matrix $\boldsymbol{A}$ for various cut-off degrees $j_{max} = j_{min} + \Delta j, \Delta j = 300, 500, \ldots, 1600$, and a body with axisymmetric surface generated by height $H(\vartheta, \lambda = 80°)$ ($e_0^2 = 0$ and $j_{min} = 10\,000$)

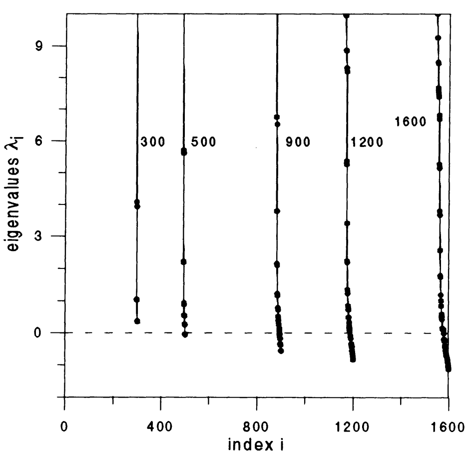

Fig. 6a. A detail of Fig. 6

degree j_{zero} for which the existence of the solution of the Stokes pseudo-boundary-value problem is not guaranteed can be estimated by analysing the existence of a solution for a model with a constant topographical heights over the world. If we replace H_0 in the example in section 7.1. with the maximum topographical height H_{max}, then such an estimate j_{const} obviously underestimates the actual j_{zero}, i.e., it is too pessimistic, and hence it holds

$$j_{zero} \geq j_{const} \ , \tag{46}$$

where j_{const} is determined by the roots of function $K_j(H_{max})$ given by eqn. (42), i.e., j_{const} satisfies the equation

$$(j_{const} + 1)\left(\frac{R}{R + H_{max}}\right)^{j_{const}+2} - 2 = 0 \ . \tag{47}$$

For the examples in Fig. 4 and 6, we obtain $j_{const} \doteq 698$ when $H_{max} = 53\,530$ metres, and $j_{const} \doteq 10\,158$ when $H_{max} = 5353$ metres. We have already learnt that the actual critical numbers are $j_{zero} \doteq 800$ and $j_{zero} \doteq 10\,500$, respectively. So, the criterion (46) estimates j_{zero} quite well.

8. Different approximations leading to the fundamental equation of physical geodesy

There are at least three approximate approaches to convert the Stokes pseudo-boundary-value problem to the regular Stokes problem. All of them try to express the 'two-boundary' condition (31) in terms of a condition referred to one boundary only, the Earth's surface or the geoid. A common and also an easy way, used, e.g., by Vaníček and Kleusberg (1987), is based on the belief that the approximation

$$\left.\frac{\partial T^h}{\partial r}\right|_P \doteq \left.\frac{\partial T^h}{\partial r}\right|_{P_g} \tag{48}$$

does not generate large errors in the resulting geoidal heights. The solution of the Stokes problem can then readily to be found by employing Stokes's integration (Heiskanen and Moritz, 1967, Sect. 2-16.). However, Vaníček et al. (1995) showed that the approximation (48) may cause systematic errors in geoidal heights in magnitudes of several decimetres.

The second possibility consists of developing the radial derivative of potential T^h into a Taylor series:

$$\left.\frac{\partial T^h}{\partial r}\right|_P = \left.\frac{\partial T^h}{\partial r}\right|_{P_g} + \left.\frac{\partial^2 T^h}{\partial r^2}\right|_{P_g} H + \ldots \tag{49}$$

Taking only the first two terms of this Taylor series expansion, and putting approximately $\epsilon_h(T_P^h) \doteq \epsilon_h(T_{P_g}^h)$, boundary condition (31) takes the form

$$\left.\frac{\partial T^h}{\partial r}\right|_{P_g} + \frac{2}{r_Q} T_{P_g}^h = -\Delta g^h - g_1 + \epsilon_h(T_{P_g}^h) + \epsilon_\gamma(T_{P_g}^h) \ , \tag{50}$$

where

$$g_1 = \left.\frac{\partial^2 T^h}{\partial r^2}\right|_{P_g} H \ . \tag{51}$$

Now, the two requirements of Stokes's integration are satisfied: the boundary condition (50) is referred to a point on the geoid, and the disturbing potential T^h is harmonic outside the geoid. Therefore, Stokes's integration may be immediately applied to eqn. (50).

However, the right-hand side of eqn. (50) contains the term g_1 which makes the problem difficult. Since $\partial^2 T^h / \partial r^2$ is the vertical gradient of the anomalous gravitation $\partial T^h / \partial r$, the term $g_1 = (\partial^2 T_{P_g}^h / \partial r^2) H$ represents the harmonic downward continuation of the anomalous gravitation from P to P_g. Since the actual gravity field is not known in topographical masses, the term g_1 can be evaluated only approximately or the boundary-value problem formulated above must be solved iteratively starting with some model for g_1. The latter approach leads, in fact, to the downward continuation of gravity from the Earth's surface to the geoid. Such a procedure is unstable and requires some kind of regularization to suppress amplifications of short wavelengths, however, there are then no problems with the existence of a solution. Hence, approximation (49) changes the nature of the problem—it requires a more or less sophisticated procedure in preparing the right-hand side of the boundary condition (50) from the observed data instead of dealing with the solvability of the boundary-value problem formulated for original observations; the possibility of the non-existence of the solution is lost by this approximation.

It should be noted that Molodenskij et al. (1960) and later Moritz (1980, Sect. 45) made the suggestion to evaluate the term g_1 only synthetically in order to avoid the problem with the non-stability of the downward continuation procedure. They assume that the gravity anomalies are linearly dependent on topographical heights and the Poisson integration for the term g_1 is taken over topographical heights and not over gravity data. However, the linear relationship between free-air gravity anomalies and topographical heights introduced by Pellinen (1962) holds only approximately (Heiskanen and Moritz, 1967, Figure 7-6). The question as to the errors of geoidal heights due to this approximation still remains open. The main advantage of the approximation (50) is that the solution of the problem (14), (17) and (50) in the spherical approximation may easily be expressed by means of Stokes's integration.

Perhaps the least drastic approximation is to refer the second term on the left-hand side of eqn.(31) to the Earth's surface. Formally, boundary condition (31) may be rewritten in the form

$$\frac{\partial T^h}{\partial r}\bigg|_P + \frac{2}{r_P} T^h\bigg|_P = -\Delta g^h - DT^h + \epsilon_h(T_P^h) + \epsilon_\gamma(T_{P_g}^h) \ , \qquad (52)$$

where

$$DT^h = \frac{2}{r_g} T^h\bigg|_{P_g} - \frac{2}{r_P} T^h\bigg|_P \ . \qquad (53)$$

Let us make an estimate of the maximum of DT^h,

$$|DT^h| \doteq \frac{2}{r_P}\left|\frac{\partial T^h}{\partial r}\right|_P H \le 200\frac{H}{r_P}\mathrm{mGal} \le 0.25 \ \mathrm{mGal} \ , \qquad (54)$$

where the gravity disturbances $\partial T^h/\partial r$ on the Earth's surface have been estimated by the value of 100 mGal, and the height H of the Earth's surface above the geoid by 8900 metres. In most practical applications, term $DT^{h,\ell}$ may be neglected because its maximum size is less than the accuracy of gravity data available for geoid determination. When the accuracy of gravity anomalies is better than 0.25 mGal, then the term DT^h may be computed from existing models of the geoid, or the Stokes problem with approximate boundary condition (52) may be solved iteratively starting with $DT^h = 0$, and improving it successively.

One possible way to solve this problem consists of two steps (Vaníček et al., 1995; Martinec, 1996). First, harmonic function $r\partial T^h/\partial r + 2T^h$ is continued from the Earth's surface to the geoid, and then Stokes's integral is employed to find the potential T^h on the geoid. The downward continuation of a harmonic function is an unstable procedure meaning that the solution exponentially diverges at infinite frequency. (To get a bounded, non-oscillating solution, some kind of regularization must again be applied to the problem.) Once the boundary condition (31) is approximated by the boundary condition (52) and the problem is solved as outlined, the property of the problem on the existence is again changed. Whereas the solution of the original Stokes pseudo-boundary-value problem does not exist for a finite geoidal wavelength, the solution of the approximate problem with boundary condition (52) exists except for the geoidal wavelength of an infinitesimally short wavelength that must be completely suppressed prior to downward continuation.

It is possible to avoid the downward continuation and to deal directly with (52). The main disadvantage of this approach is that the boundary operator is referred to the Earth's surface which cannot be approximated by a smooth boundary. That is why a simple analytical formula solving (14), (17), (52) is not available. Nevertheless, the nature of the problem (14), (17), (52) is again different from the nature of the original Stokes pseudo-boundary-value problem. Figure 7 demonstrates this fact in a transparent way. For the axisymmetric example shown in Fig. 6, we compare the eigenvalue spectrum of matrix $\boldsymbol{A}$ for the original Stokes pseudo-boundary-value problem to that of the Stokes problem with approximate condition (52). Whereas the first eigenvalue spectrum intersects the zero level at harmonic degree $j_{zero} \doteq$ 10 500, the second spectrum approaches zero for $j \to \infty$.

9. Conclusions

This paper formulated and discussed the existence of a solution to the Stokes pseudo-boundary-value problem for geoid determination. We derived the boundary condition (31) relating to this problem without assuming that the surface gravity data had been continued from the Earth's surface to the geoid. The boundary condition (31) has an unusual form, because it contains the

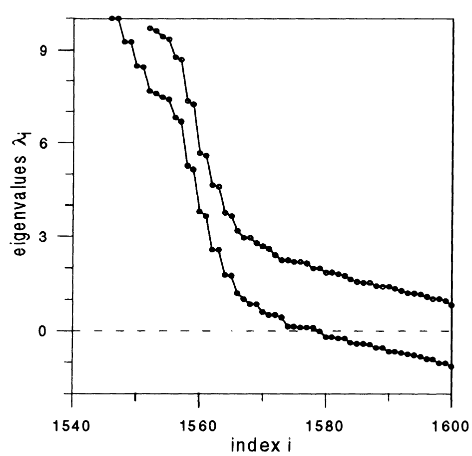

Fig. 7. The eigenvalue spectra of matrix $\boldsymbol{A}$ for boundary condition (31) (lower branch), and approximate boundary condition (52) (upper branch) (the body is the same as that considered in Fig. 6; $j_{min} = 10\,000$, and $j_{max} = 11\,600$)

unknown anomalous potential referred to both the Earth's surface and the geoid coupled by the known topographical height. The numerical analysis of the 'two-boundary' condition (31) performed for a simplified model of the Earth's surface has revealed that the transfer matrix between the unknown potential on the geoid and the surface gravity anomalies may become ill-conditioned or even singular at a certain critical wavelength of a *finite* length. The existence of a solution is not guaranteed for this critical geoidal wavelength. Once this ill-posed case occurs, to obtain a bounded and non-oscillating solution, the Stokes pseudo-boundary-value problem must be regularized in such a way that this critical geoidal wavelength and its vicinity are excluded from the solution. We have given an estimate of the critical geoidal wavelength; for the highest part of the Earth's surface, the critical geoidal wavelength is about 1 arcmin.

Furthermore, we discussed three possibilities of transforming the original form of a 'two-boundary' condition into a 'one-boundary' condition relating to the geoid or the Earth's surface. The approach, based on Taylor's series expansion of the radial derivative of unknown potential, leads to the downward continuation of a harmonic function from the Earth's surface to the geoid the solution of which is unstable for an infinitesimally short geoidal wavelength. It means that the domain of solvability of Stokes pseudo-boundary-value problem is changed by the Taylor series expansion; the solution becomes unstable for shorter geoidal wavelengths compared to that of the original Stokes pseudo-boundary-value problem.

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